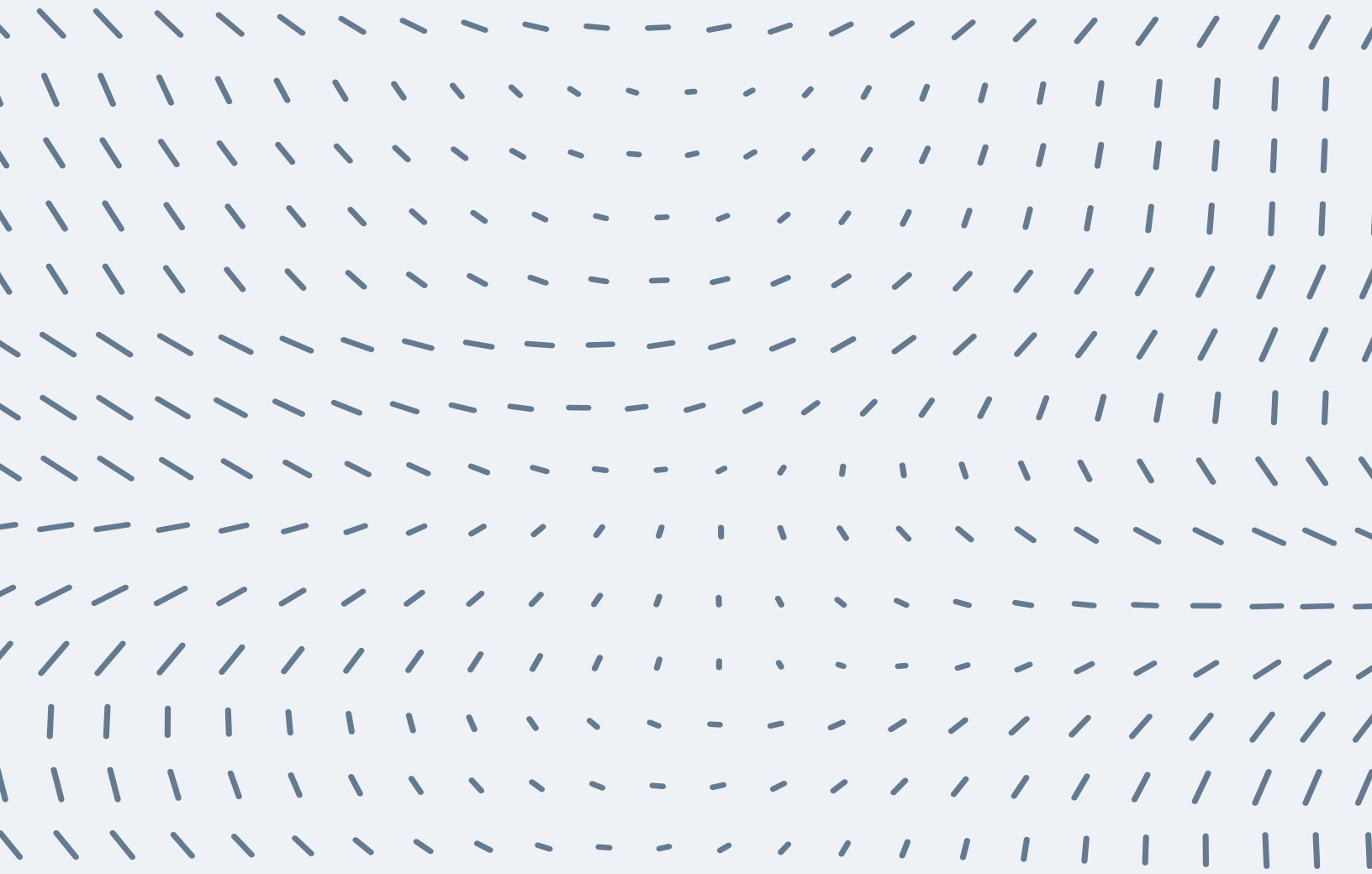
Applied Ordinary Differential Equations

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APMA 0350 (Fall 2021)

§1 Introduction

Motivation: differential equations are used to model and describe many processes in

- biology (epidemiology, ecology, cancer research, cell + developmental biology, ...)
- chemistry (reaction networks)
- climate research + weather forecasts
- economics (option pricing, wealth + income distribution)
- engineering + physics (from mass-spring systems to aircraft design)
- neuroscience + computer science (models of brain activity patterns, deep learning networks)

Learning goals:

- design differential-equation models
- analyse differential equations using
  - explicit analytic solutions
  - geometric qualitative theory
  - numerical simulations
- interpret solutions to differential equations
- understand theory of differential equations
  - existence and uniqueness of solutions

Differential equations:

> A differential equation is an equation that contains one or more derivatives (of quantities we are interested in)

- Example 1: $\underbrace{\frac{dy}{dt}}_{\text{derivative}} = -ay$ → $y = y(t)$ is a function of t
  - → y dependent variable
    t independent variable

Solutions: A function $y = \varphi(t)$ is a solution of $\frac{dy}{dt} = -ay$ if $\frac{d\varphi(t)}{dt} = -a\varphi(t)$ for all values of t in an open nonempty interval

Examples:

- Is $y(t) = t^2$ a solution of $\frac{dy}{dt} = -ay$?

  Check: $\frac{dy(t)}{dt} = \frac{d}{dt}(t^2) = 2t$
  $-ay(t) = -at^2$
  } not equal (except at $t=0$ and $t=-\frac{2}{a}$ which are isolated points)

- Is $y(t) = e^{-at}$ a solution to $\frac{dy}{dt} = -ay$?

  Check: $\frac{dy(t)}{dt} = \frac{d}{dt}(e^{-at}) = -ae^{-at}$
  $-ay(t) = -ae^{-at}$
  } equal for all t

  → $y(t) = e^{-at}$ is a solution of $\frac{dy}{dt} = -ay$

- Example 2: $\frac{d^2u}{dx\,dt} + xu^2 = \left(\frac{du}{dx}\right)^2$ (derivatives)

  } u dependent variable; x,t independent variables

  → seek solutions in the form $u(x,t)$

- Example 3:
  $$\begin{cases} -y + \frac{dy}{dt} - \frac{d^2x}{dt^2} = 0 \\ x + \frac{d^2y}{dt^2} = 0 \end{cases}$$

  } this is a differential equation ✓

  } x,y dependent variables; t independent variable

  → seek solutions in the form $x(t), y(t)$

A solution to a differential equation is a function that depends on the independent variables and, when substituted for the dependent variables in the differential equation, gives an identity (that is, left-hand side = right-hand side) for all values of the independent variables.

Classification of differential equations:

1. Identify dependent and independent variables
2. Number of dependent variables
   - $= 1$ scalar differential equation
   - > 1 system of differential equations
3. Number of independent variables
   - $= 1$ ordinary differential equation (ODE)
   - > 1 partial differential equation (PDE)
4. Order of a differential equation: highest derivative that appears
5. Linear or nonlinear:

   Linear: whenever the dependent variables or any of their derivatives appear in a term of the differential equation, they must occur alone with power 1

   nonlinear: otherwise

Examples of linear versus nonlinear:

- assume that
  - u, v dependent variables
  - x, t independent variables

- $3v^4$ — nonlinear: v raised to power 4
- $\left(\frac{dv}{dx}\right)^2$ — nonlinear: $\frac{dv}{dx}$ raised to power 2
- $v\frac{dv}{dt}$ — nonlinear: v and $\frac{dv}{dt}$ each have power 1, but we multiply them to get an effective power 2
- $t^2 v$ — linear: v has power 1 (t^2 involves the independent variable which does not matter)
- $\sin(v)$ — nonlinear: $\sin(v_1 + v_2) \neq \sin v_1 + \sin v_2$
- $\frac{1}{v}$ — nonlinear: $\frac{1}{v} = v^{-1}$ has power -1
- $\frac{dv}{dx} v$ — nonlinear: similar to $v\frac{dv}{dt}$ term above, but now involving two different dependent variables
- $10e^x$ — linear: no dependent variable occurs

- To determine linear vs nonlinear, temporarily ignore the independent variables, differential symbols, and any constants; check whether the resulting terms are linear in the dependent variables.

Classify the following differential equations:

(1) $\frac{d^4x}{dt^4} = t^2 + x e^t$

- x dependent → scalar
- t independent → ODE
- order 4
- linear

(2) $\frac{dy}{dx} + x^2 y = \frac{1}{y}$

- y dependent → scalar
- x independent → ODE
- order 1
- nonlinear ($\frac{1}{y}$ term)

(3) $\frac{dx}{dt} + \frac{d^2x}{dt\,dy} = x + t^2$

- x dependent → scalar
- t, y independent → PDE
- order 2 ($\frac{d^2x}{dt\,dy}$)
- linear

(4) $\begin{cases} \frac{dx}{dt} = xy \\ \frac{dy}{dt} = -3y \end{cases}$

- x, y dependent → system
- t independent → ODE
- order 1
- nonlinear ($x \cdot y$)

<u>Outlook</u> In AMA 0350, we will focus on ordinary differential equations (scalar + systems; linear + nonlinear)

Partial differential equations are covered in AMA 0360.

§2 First-order scalar ODEs

We will consider first-order scalar ODEs of the form

$$\frac{dx}{dt} = f(x,t)$$

- $x \in \mathbb{R}$ dependent variable
- $t \in \mathbb{R}$ independent variable

and discuss how to

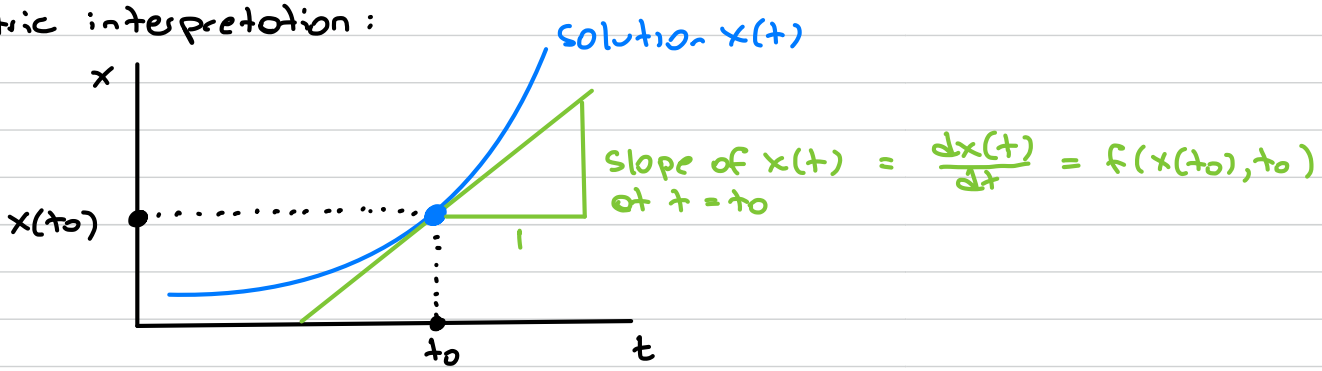
- gain insight into their solutions without finding explicit solutions
- solve certain classes of equations analytically
- develop models of real-world problems in the above form
- determine whether solutions exist and are unique

Recall that a function $x(t)$ is said to be a solution of $\frac{dx}{dt} = f(x,t)$ if

$$\frac{dx(t)}{dt} = f(x(t),t) \quad \text{for all values of } t$$

(or all values of t contained in some nonempty interval (a,b))

Geometric interpretation:



We conclude that

- $x(t)$ will increase strictly if $f(x(t),t) > 0$
- $x(t)$ will decrease strictly if $f(x(t),t) < 0$

→ these features will help us

- sketch solutions
- design models

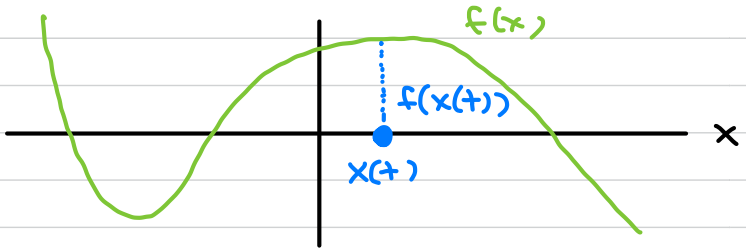
- $f(x) > 0$: x increases
- $f(x) < 0$: x decreases

§2.1 Qualitative approach

Consider the scalar differential equation

$$\frac{dx}{dt} = f(x) \qquad \text{"autonomous"} \qquad \left(\begin{array}{l}\text{no explicit dependence}\\ \text{of } f \text{ on independent}\\ \text{variable } t\end{array}\right)$$

Example :

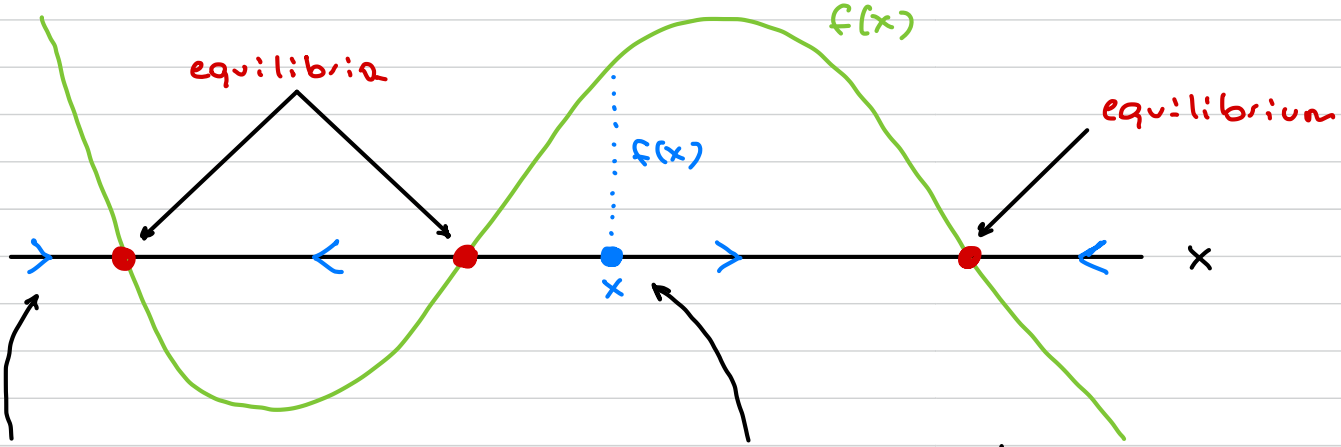


We will think of a particle moving along the x-axis.
We denote the position of the particle at time t by $x(t)$ and assume that its velocity $\frac{dx}{dt}(t)$ is given by $f(x(t))$.

$\longrightarrow$ $x(t)$ satisfies the ODE $\frac{dx}{dt} = f(x)$

or velocity at time t = f(position at time t)

Goal : given the graph of $f(x)$, understand the motion of the particle as a function of time t.



Arrows on the horizontal axis indicate direction of movement of particles as time t increases

$f(x) > 0$ hence $\frac{dx}{dt} = f(x) > 0$ hence $x(t)$ increases $\Rightarrow$ particle moves to the right

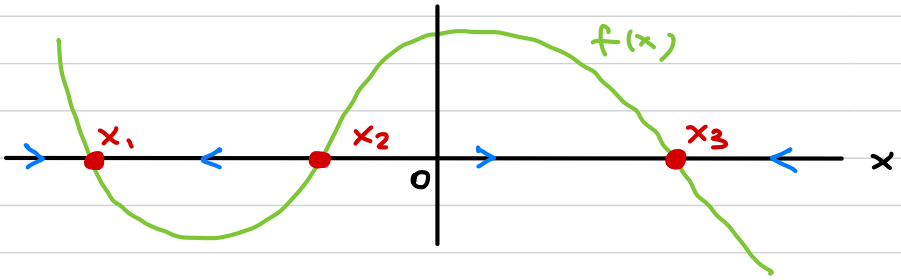
(i) $f(x) > 0 \Rightarrow \frac{dx}{dt} = f(x) > 0 \Rightarrow x(t)$ increases as t increases
→ particle moves to the right

(ii) $f(x) < 0 \Rightarrow \frac{dx}{dt} = f(x) < 0 \Rightarrow x(t)$ decreases as t increases
→ particle moves to the left

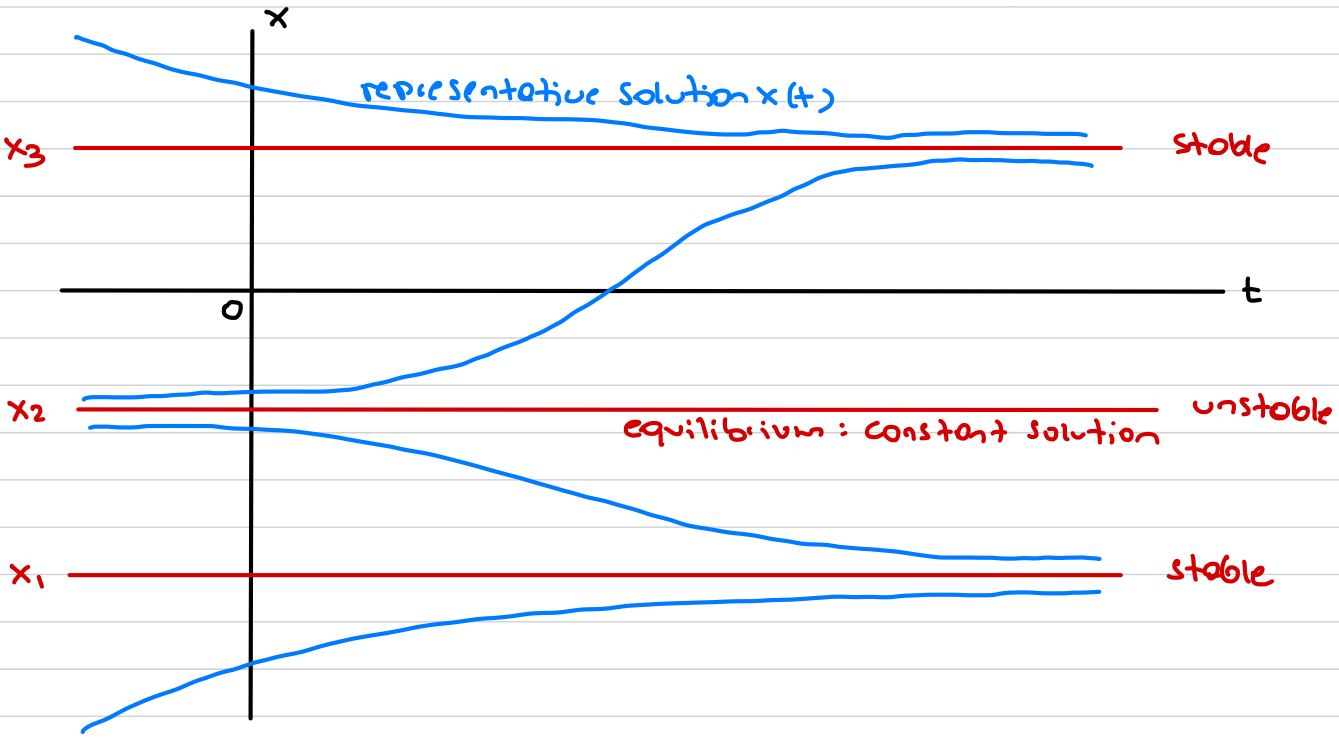
(iii) $f(x_*) = 0 \Rightarrow x(t) = x_*$ for all t is a solution!

Check: LHS $= \frac{dx(t)}{dt} \xrightarrow{x(t) = x_* = \text{constant}} 0$
RHS $= f(x(t)) = f(x_*) = 0$ ✓
} equal for all values of t

→ roots or zeros of $f(x)$ (that is, points x_* for which $f(x_*) = 0$) correspond to time-independent solutions $x(t) = x_*$ for all t
→ these are called equilibria or rest states
A particle positioned at a rest state x_* will not ever move

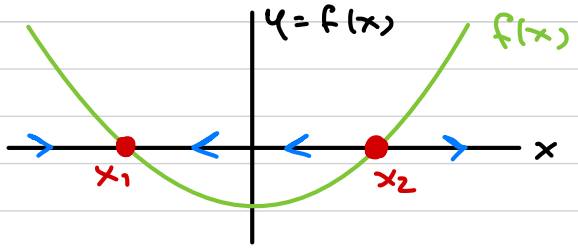


We can use the equilibria and arrows to sketch the graphs of representative solutions $x(t)$ as functions of time t:



Qualitative theory for $\frac{dx}{dt} = f(x)$ ← no dependence on the independent variable t

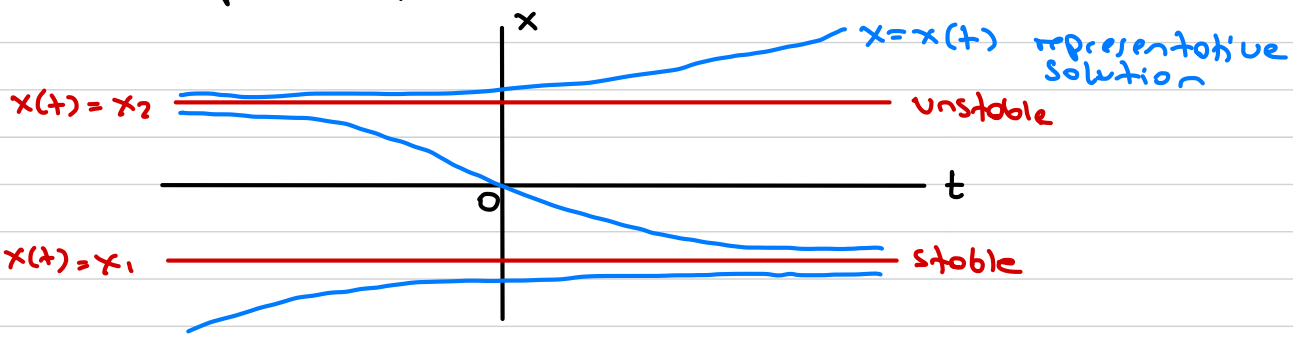
① Draw graph of right-hand side $f(x)$



x_1, x_2 are rest states
$>, <$ direction of movement

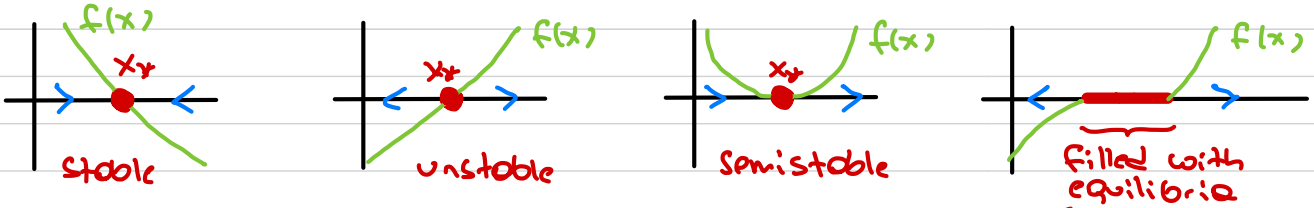
- If $f(x_*) = 0$, then $x(t) = x_*$ for all t is a solution
  → rest states or equilibria
- If $f(x) > 0$, then $x(t)$ increases strictly
  If $f(x) < 0$, then $x(t)$ decreases strictly
  → indicate direction of movement by arrows on x-axis

② Sketch representative solutions $x = x(t)$ as functions of t



note: we have many solutions which are distinguished by the initial position of the particle at $t = 0$ (see below!)

③ Stability of rest states



nearby solutions converge to x_* as $t \to \infty$

nearby solutions move away from x_* as t increases

- stable from one side
- unstable from the other side

Initial conditions and uniqueness

We can place the particle at any initial position x_0 at time $t=0$, and the motion of the particle should then be determined uniquely by the differential equation.

→ We expect that the differential equation $\frac{dx}{dt} = f(x)$ with the additional condition $x(0) = x_0$ for a given x_0 has a unique solution $x(t)$ defined for all t near 0.

Theorem Assume that $f(x,t)$ and $\frac{df}{dx}(x,t)$ are continuous for all (x,t). For each given x_0, t_0, the differential equation $\frac{dx}{dt} = f(x,t)$ with prescribed initial condition $x(t_0) = x_0$ has a unique solution $x(t)$, defined for $t_0 - \delta < t < t_0 + \delta$ for some $\delta > 0$.

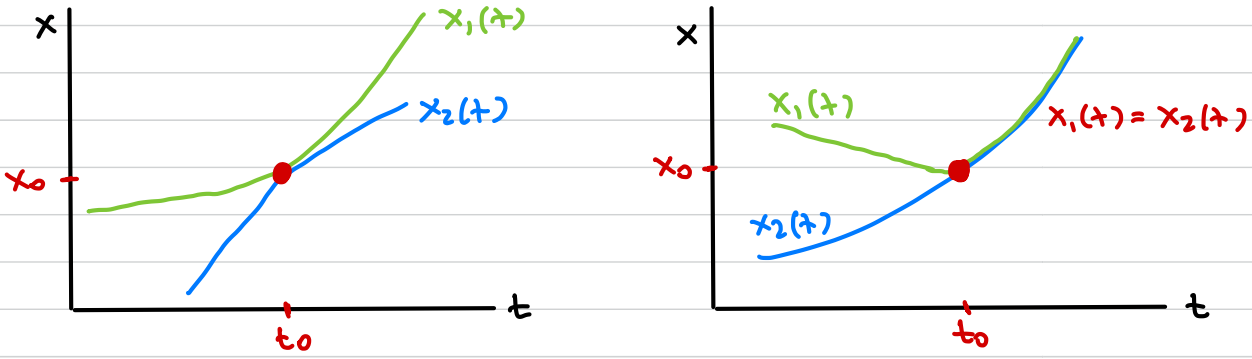
Uniqueness of solutions means the following:

If $x_1(t)$ (defined for $t_0 - \delta_1 < t < t_0 + \delta_1$) and $x_2(t)$ (defined for $t_0 - \delta_2 < t < t_0 + \delta_2$) are both solutions of

$$\begin{cases} \frac{dx}{dt} = f(x,t) \\ x(t_0) = x_0 \end{cases}$$

then $x_1(t) = x_2(t)$ for all t with $t_0 - \delta < t < t_0 + \delta$ with $\delta = \min\{\delta_1, \delta_2\}$.

→ if $f(x,t)$ and $\frac{df}{dx}(x,t)$ are continuous for all (x,t), then solution graphs of $\frac{dx}{dt} = f(x,t)$ cannot cross:



neither figure is possible if $f(x,t)$ and $\frac{df}{dx}(x,t)$ are continuous for all x,t

Examples:

① Does $\frac{dx}{dt} = (x+t)^2 \sin t$ with $x(10) = -5$ have a unique solution?

Answer: The function $f(x,t) = (x+t)^2 \sin t$ is continuous in (x,t). Furthermore, this function is differentiable in x and $\frac{\partial f}{\partial x}(x,t) = 2(x+t)\sin t$ is continuous in (x,t). Hence, the conditions of the theorem are satisfied, and the theorem ensures that the initial-value problem above has a solution and that the solution is unique.

② Does $\frac{dv}{dx} = \sqrt{v}$ with $v(1) = 0$ have a unique solution?

Answer: The dependent variable is v, and the function $f(v) = \sqrt{v}$ is not defined for $v < 0$ and $\frac{\partial f}{\partial v} = \frac{1}{2\sqrt{v}}$ is not defined for $v \leq 0$. Hence, the theorem does not apply → we cannot answer this question.

Note: The solution may exist, and it may be unique, but we cannot use the theorem to decide this.

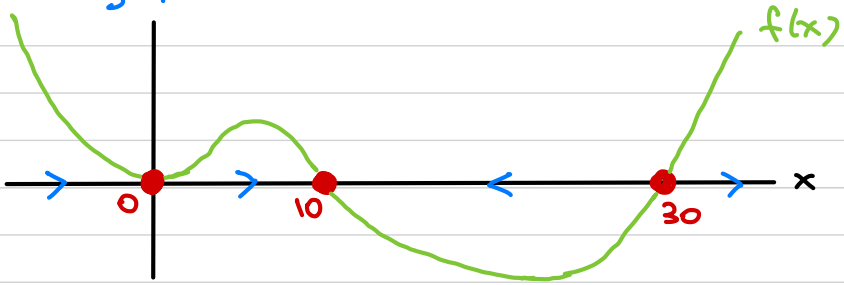
Example Carry out a qualitative analysis of $\frac{dx}{dt} = \underbrace{x^2(x-10)(x-30)}_{=: f(x)}$

① Classify and check existence & uniqueness
- first-order, scalar, autonomous ODE → qualitative approach applicable ✓
- $f(x)$, $\frac{df}{dx}(x)$ continuous for all x → solutions exist and are unique ✓

② Find zeros of $f(x)$

$f(x) = x^2(x-10)(x-30) = 0$ for $x = 0, 10, 30$: equilibria

③ Plot graph of $f(x)$



④ Draw arrows on horizontal axis to indicate direction of movement

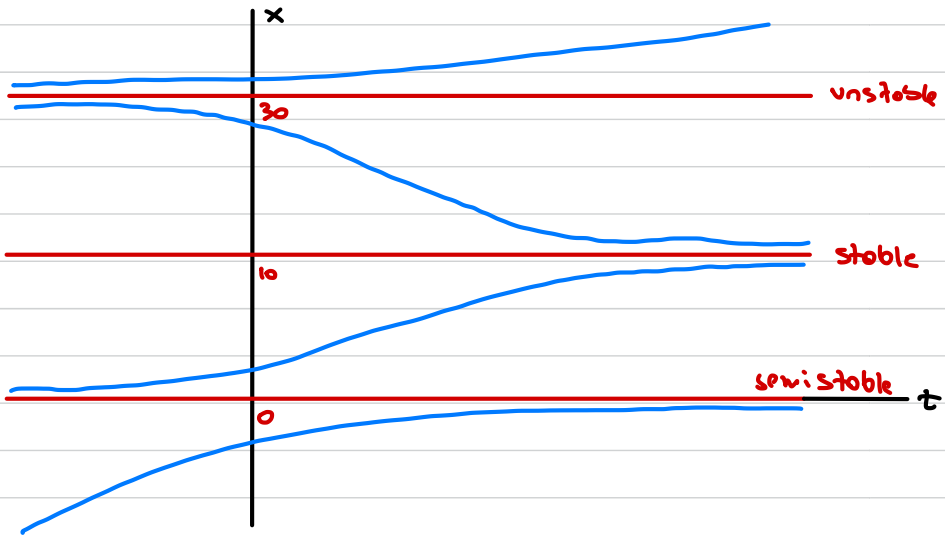
→ where $f(x) > 0$

← where $f(x) < 0$

⑤ Determine stability of equilibria

$x = 0$	semistable
$x = 10$	stable
$x = 30$	unstable

⑥ Sketch graphs of representative solutions

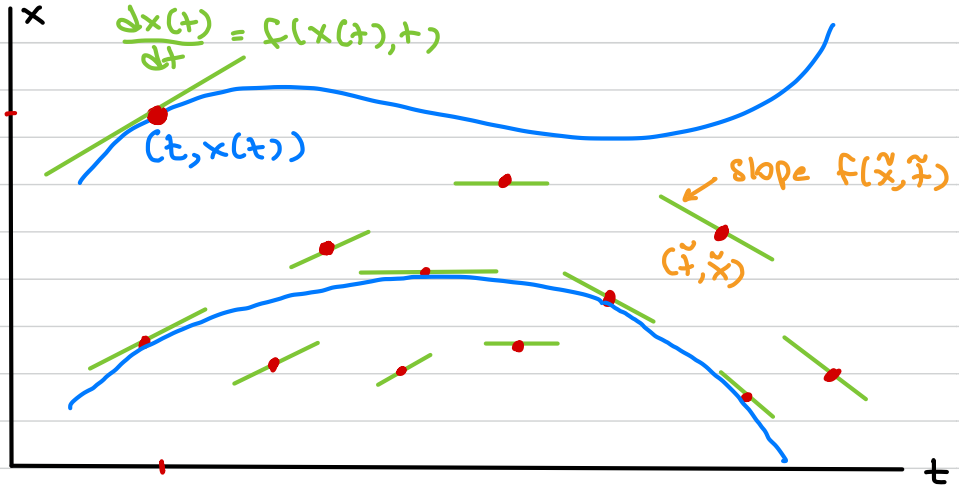


Direction fields for non-autonomous scalar ODEs

Consider $\frac{dx}{dt} = f(x,t)$: right-hand side may now depend on the independent variable t

- our qualitative approach will not work since $f(x,t)$ now depends on both x and t.
- we can still use the geometric idea that $f(x(t),t)$ is the slope of the solution $x(t)$ at time t.

Recall that $x(t)$ is a solution of $\frac{dx}{dt} = f(x,t)$ if $\underbrace{\frac{dx(t)}{dt}}_{\text{slope of } x(t) \text{ at time } t} = f(x(t),t)$ for all t



„direction field"

→
- Solutions $(t,x(t))$ through a point $(\tilde{t},\tilde{x})$ must be tangent to the line with slope $f(\tilde{x},\tilde{t})$ through $(\tilde{t},\tilde{x})$.
- If $f(x,t)$ and $\frac{df}{dx}(x,t)$ are continuous for all (x,t), solutions cannot touch or cross.

Summary

① Solutions: $x(t)$ is a solution of $\frac{dx}{dt} = f(x,t)$ if $\frac{dx(t)}{dt} = f(x(t),t)$ for all $t \in (0,b)$ for some $0<b$

② classification: allows us to decide which method to use and what behavior of solutions to expect

③ existence and uniqueness: ascertain when we know that an ODE has a solution and under which condition solutions are unique

④ qualitative analysis: gives insights into shape of solutions and their behavior as $t \to \infty$

note: this approach builds on, and requires, existence and uniqueness since we otherwise do not know whether a solution even exists!

⑤ direction fields: gives insights into solutions for nonautonomous scalar ODEs; some caveat as under ④!

§2.2 Separable equations

Goal: develop methods to find explicit solutions to scalar ODEs.

→ we focus on separable equations for which we can collect dependent and independent variables on different sides of the equation

Definition: A scalar first-order ODE is said to be separable if it can be brought into the form

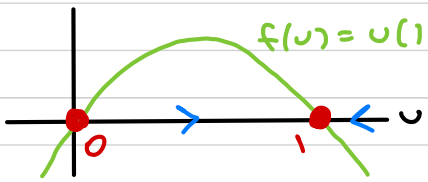
$$F(x)\frac{dx}{dt} = G(t) \qquad \text{(or „}F(x)\,dx = G(t)\,dt\text{")}$$

these functions may be constants

Outline:
- find practical informal way to solve separable ODEs
- justify this approach using the chain rule

Example 1: Solve $\frac{du}{dt} = u(1-u)$ with $u(0) = u_0 \in (0,1)$.

recall



so expect $u(t) \to 1$ as $t \to \infty$

- separable? yes: $\underbrace{\frac{1}{u(1-u)}}_{\text{depends only on } u} \frac{du}{dt} = \underbrace{1}_{\text{depends only on } t}$ ✓
- formally: $\frac{1}{u(1-u)}\, du = 1 \cdot dt$
- integrate: $\int_{u_0}^{u} \frac{1}{u(1-u)}\, du = \int_0^t 1 \cdot dt$

since $u = u_0$ at $t = 0$; we seek solution u at time t

$\to \int_{u_0}^{u} \frac{1}{u(1-u)}\, du = \int_0^t 1\, dt$ partial fractions

$$\int_{u_0}^{u} \left(\frac{1}{u} + \frac{1}{1-u}\right) du = t$$

$$\ln \frac{u}{u_0} - \ln \frac{1-u}{1-u_0} = t$$

$$\ln \left(\frac{u(1-u_0)}{u_0(1-u)} \right) = t$$

$$\frac{u(1-u_0)}{u_0(1-u)} = e^t$$

$\to$ $$u(1-u_0) = u_0(1-u)e^t$$

$$u(1-u_0+u_0e^t) = u_0e^t$$

- Solution: $$u(t) = \frac{u_0 e^t}{1 + u_0(e^t - 1)}$$

- Since $f(u) = u(1-u)$ is continuously differentiable in u, our solution is the unique solution to

$$\frac{du}{dt} = u(1-u) \quad \text{with} \quad u(0) = u_0 \quad \text{where} \quad 0 < u_0 < 1.$$

- Note: $v(t) = \dfrac{v_0 e^t}{1 + v_0(e^t - 1)} \longrightarrow \begin{cases} 1 & \text{as } t \to \infty \\ 0 & \text{as } t \to -\infty \end{cases}$

as expected from qualitative analysis

Example 2: Solve $\dfrac{dy}{dx} = y(x+y)$

- separable? NO: cannot write this in the form $\dfrac{G(x)}{F(y)}$
  → cannot use this approach to find a solution

Example 3: Solve $\dfrac{dy}{dx} = \dfrac{y(2x+\cos x)}{1+y^2}$ with $y(0) = 1$

- separable? yes: $\dfrac{1+y^2}{y} \cdot \dfrac{dy}{dx} = 2x + \cos x$ ✓
- formally: $\left(\frac{1}{y} + y\right) dy = (2x + \cos x)\, dx$
- option 1: proceed as above:

$$\int_1^y \left(\frac{1}{y} + y\right) dy = \int_0^x (2x + \cos x)\, dx$$

because $y(0) = 1$ means $y = 1$ at $x = 0$

→ evaluate integrals ...

- option 2: integrate using indefinite integrals.

$$\int \left(\frac{1}{y} + y\right) dy = \int (2x + \cos x)\, dx$$

$$\ln|y| + \frac{y^2}{2} = c + x^2 + \sin x$$

cannot be solved for $y = y(x)$ as a function of x

- general solution (in implicit form):

$$\ln|y| + \frac{y^2}{2} = c + x^2 + \sin x$$

where c is an arbitrary constant

- initial condition: $y(0) = 1$ or $y = 1$ when $x = 0$

$\rightarrow \quad \ln 1 + \frac{1^2}{2} = C + 0^2 + \sin 0 \quad$ gives $C = \frac{1}{2}$

$\ln|y| + \frac{y^2}{2} = \frac{1}{2} + x^2 + \sin x$
gives the solution with $y(0) = 1$ in implicit form

Procedure for solving separable equations

- Separate variables: $F(x)\frac{dx}{dt} = G(t)$
- definite integral: $\int_{x_0}^{x} F(x)\,dx = \int_{t_0}^{t} G(t)\,dt$ — solution will satisfy $x(t_0) = x_0$
- indefinite integral: $\int F(x)\,dx = \int G(t)\,dt$ — general solution
  integration constant C can be used to satisfy initial conditions if specified
- implicit form: typically, solution will be given only implicitly; sometimes, we can solve for x as a function of t

Justification:

Assume $x(t)$ is differentiable and satisfies

$$\int_{x_0}^{x(t)} F(x)\,dx = \int_{t_0}^{t} G(t)\,dt \quad \text{for all } t \in (a,b)$$

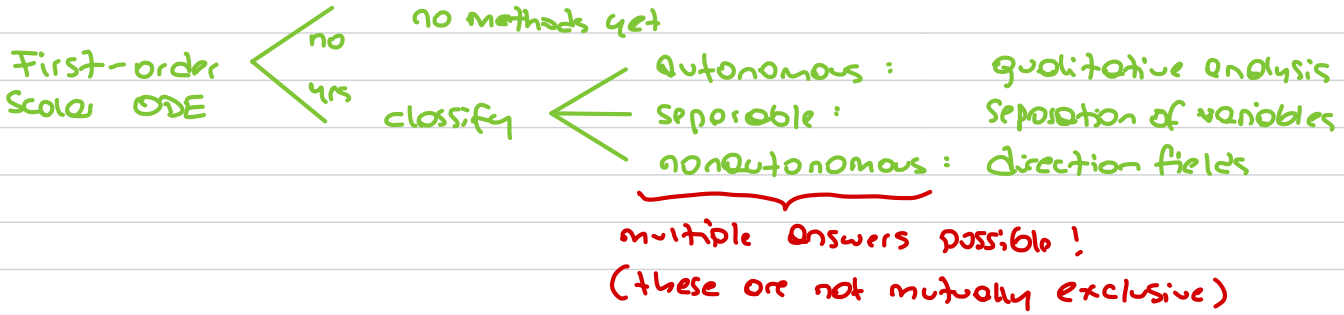
Differentiating in t gives

$$F(x(t))\frac{dx(t)}{dt} = G(t) \quad \text{for all } t \in (a,b)$$

and $x(t)$ is a solution of $F(x)\frac{dx}{dt} = G(t)$ for $t \in (a,b)$ ✓

Develop your own personal decision tree

Decide which methods apply to a given ODE:



§2.3 Linear first-order scalar ODEs

The general form of a linear scalar first-order ODE (with dependent variable x and independent variable t) is

$$(1) \qquad a_0(t)\frac{dx}{dt} + a_1(t)\,x + a_2(t) = 0$$

We will always bring equations into the standard form given by

$$(2) \qquad \frac{dx}{dt} = a(t)\,x + b(t) \qquad : \text{ standard form}$$

(divide (1) by $a_0(t)$ and collect all terms except $\frac{dx}{dt}$ on the right-hand side)

Goal: derive a formula for solutions of (2)
(note: (2) is not separable unless $b(t) = 0$ for all t)

Step 0 Existence and uniqueness for (2)

We know that $\frac{dx}{dt} = f(x,t)$ with $x(t_0) = x_0$ for given x_0, t_0 has a unique solution provided $f(x,t)$ and $\frac{\partial f}{\partial x}(x,t)$ are both continuous in (x,t) for all x,t.

We have $f(x,t) = a(t)x + b(t)$ and $\frac{\partial f}{\partial x}(x,t) = a(t)$. Hence, if $a(t)$ and $b(t)$ are continuous for all t, then for each x_0, t_0 the ODE (2) with the initial condition $x(t_0) = x_0$ has a unique solution defined for $t_0 - \delta < t < t_0 + \delta$ for some $\delta > 0$.

⟶ Assume $a(t)$ and $b(t)$ are continuous for all t

Step 1: Homogeneous equation $\frac{dx}{dt} = a(t)x$

(3) $\frac{dx}{dt} = a(t)x$ separable → use separation of variables

- Note that $x(t) = 0$ for all t is a solution of (3) and uniqueness of solutions established in Step 0 shows that solutions $x(t)$ of (3) can therefore not change sign

- Separate variables and integrate:

$$\frac{1}{x}\frac{dx}{dt} = a(t)$$

$$\int_{x_0}^{x} \frac{1}{x}\,dx = \int_{t_0}^{t} a(t)\,dt$$

$$\ln\frac{x}{x_0} = \int_{t_0}^{t} a(t)\,dt$$

→ $$x(t) = x_0\, e^{\int_{t_0}^{t} a(s)\,ds}$$

absolute values not needed since solutions cannot change sign

is the unique solution of $\frac{dx}{dt} = a(t)x$ with $x(t_0) = x_0$

it is best to use different variables for the integration variable to avoid possible confusion

- Summary: The function $x(t) = x_0\, e^{\int_{t_0}^{t} a(s)\,ds}$ satisfies $\frac{dx}{dt} = a(t)x$ with $x(t_0) = x_0$ for all values of t.

  Alternatively, the general solution $x(t)$ (that is, the expression capturing all possible solutions) of $\frac{dx}{dt} = a(t)x$ is given by $x(t) = c\, e^{\int^{t} a(s)\,ds}$ where the constant c is arbitrary and can be used to satisfy any initial conditions added to the ODE.

- Example: Find the solution to $\frac{dx}{dt} + tx = 0$ with $x(1) = 2$.

  - standard form (!): $\frac{dx}{dt} = -tx$ so $a(t) = -t$.
  - solution: $x(t) = x_0\, e^{\int_{t_0}^{t} a(s)\,ds} = 2\, e^{\int_{1}^{t} (-s)\,ds}$

$$= 2\, e^{-\frac{s^2}{2}\Big|_{s=1}^{s=t}} = 2\, e^{-t^2/2 + 1/2}$$

→ $x(t) = 2\sqrt{e}\, e^{-t^2/2}$ satisfies $\frac{dx}{dt} + tx = 0$ with $x(1) = 2$

Step 2: Inhomogeneous equation $\frac{dx}{dt} = a(t)x + b(t)$

$$\frac{dx}{dt} = a(t)x + b(t) \tag{4}$$

- Recall that $x(t) = c\, e^{\int^t a(s)\,ds}$ is the solution to $\frac{dx}{dt} = a(t)x$.
- We seek solutions of (4) in the form

$$x(t) = y(t)\, e^{\int^t a(s)\,ds} \tag{5}$$

by finding an equation for $y(t)$ assuming $x(t)$ satisfies (4)

- Substituting $x(t) = y(t)\, e^{\int^t a(s)\,ds}$ into $\frac{dx}{dt} = a(t)x + b(t)$ gives

$$\text{LHS} = \frac{dx(t)}{dt} = \frac{d}{dt}\left(y(t)\, e^{\int^t a(s)\,ds}\right)$$

$$= \frac{dy(t)}{dt}\, e^{\int^t a(s)\,ds} + y(t)\, a(t)\, e^{\int^t a(s)\,ds}$$

$$\text{RHS} = a(t)x(t) + b(t) = a(t)\, y(t)\, e^{\int^t a(s)\,ds} + b(t)$$

Hence $x(t)$ satisfies $\frac{dx}{dt} = a(t)x + b(t)$ if and only if

$$\underbrace{\frac{dy(t)}{dt}\, e^{\int^t a(s)\,ds} + y(t)\, a(t)\, e^{\int^t a(s)\,ds}}_{\text{LHS}} = \underbrace{a(t)\, y(t)\, e^{\int^t a(s)\,ds} + b(t)}_{\text{RHS}}$$

cancel!

and therefore

$$\underbrace{\frac{dy(t)}{dt} = b(t)\, e^{-\int^t a(s)\,ds}}_{\text{separable!}} \tag{6}$$

- Integrating (6) in t gives

$$y(t) = c + \int^t e^{-\int^r a(s)\,ds}\, b(r)\, dr$$

and substituting this expression back into our formula (5) we see that

$$x(t) = e^{\int^t a(s)\,ds}\left[c + \int^t e^{-\int^r a(s)\,ds} b(r)\,dr\right]$$

Summary: For each constant c, the function

$$x(t) = e^{\int^t a(s)\,ds}\left[c + \int^t e^{-\int^r a(s)\,ds} b(r)\,dr\right]$$

is a solution of $\frac{dx}{dt} = a(t)x + b(t)$. The constant c can be chosen to satisfy any given initial condition.

Note: We do not even need to rederive the formula for $x(t)$: from now on, we can simply substitute specific functions $a(t)$ and $b(t)$ and evaluate the integrals to solve linear scalar first-order ODEs!

Procedure for solving linear scalar first-order ODEs:

1. Standard form: $\frac{dx}{dt} = a(t)x + b(t)$
2. Integrating factor: $\mu(t) = e^{-\int^t a(s)\,ds}$
3. General solution: $x(t) = \frac{1}{\mu(t)}\left[c + \int^t \mu(s)\,b(s)\,ds\right]$

where c is an arbitrary constant
(works also for homogeneous case $b(t) = 0$)

Common mistakes
- equation not in standard form
- parentheses missing in solution formula

Example: Solve $2\frac{dx}{dt} - te^t = \frac{2x}{t}$ with $x(1) = 0$

- Linear equation: ✓
- Existence & uniqueness: coefficients are continuous for $t > 0$ and initial condition posed at $t = 1$.
- standard form: $\frac{dx}{dt} = \underbrace{\frac{1}{t}}_{a(t)=\frac{1}{t}}x + \underbrace{\frac{t}{2}e^t}_{b(t)=\frac{t}{2}e^t}$

- Integrating factor:

$$\mu(t) = e^{-\int^t a(s)\,ds} = e^{-\int^t \frac{1}{s}\,ds} = \underbrace{e^{-\ln t}}_{\text{focus on } t>0} = \frac{1}{t}$$

- General solution:

$$\begin{aligned} x(t) &= \frac{1}{\mu(t)}\left[c + \int^t \mu(s)\,b(s)\,ds\right] \\ &= t\left[c + \int^t \frac{1}{s}\,\frac{s}{2}e^s\,ds\right] \\ &= t\left[c + \frac{1}{2}\int^t e^s\,ds\right] \\ &= t\left[c + \frac{1}{2}e^t\right] \end{aligned}$$

Hence, $x(t) = ct + \frac{t}{2}e^t$ is the general solution for $t>0$.

- Initial condition: $x(1) = 0$ gives

$$x(1) = c + \frac{e}{2} = 0 \quad \text{hence} \quad c = -\frac{e}{2}$$

- Summary:

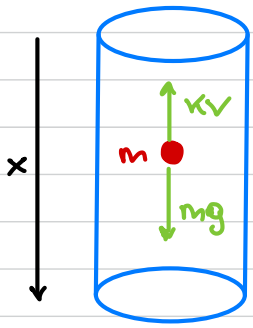
$x(t) = \frac{t}{2}(e^t - e)$ is the unique solution to $2\frac{dx}{dt} - te^t = \frac{2x}{t}$ with $x(1) = 0$

§2.4 Applications

Modelling real-world processes should start with a concrete research question we want to answer. We then develop a model that can help us answer the question — usually, the design and complexity of the model depends on the question.

Example 1 Mass falling in resistive medium

Question: A mass in a vacuum tube on the earth's surface will experience constant acceleration due to gravity and its velocity will therefore grow linearly in time. What is the velocity of a mass falling in a resistive medium such as water or honey?



Container filled with resistive medium (water, honey, ...)

forces due to
- gravity
- friction

- x — position of ball
- $v = \frac{dx}{dt}$ — velocity of ball
- $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ — acceleration of ball

Newton's law: mass × acceleration = sum of all forces acting on ball

friction: proportional to and opposite of velocity

gravity = mg
friction = $-kv$

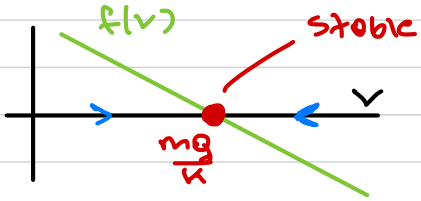
$$\rightarrow \quad m\frac{dv}{dt} = mg - kv \qquad \left(\text{or } m\frac{d^2x}{dt^2} = mg - k\frac{dx}{dt}\right)$$

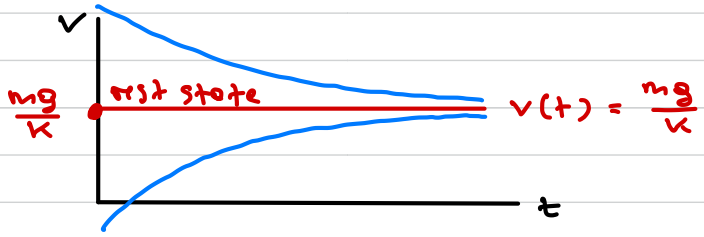
$$\rightarrow \quad \frac{dv}{dt} = g - \frac{kv}{m} =: f(v)$$

first-order, scalar, linear, autonomous, separable

Existence and uniqueness (upon adding initial condition $v(t_0) = v_0$) guaranteed by theorem since $f(v)$ is continuously differentiable in v for all v.

① Qualitative analysis





Conclusion: regardless of initial velocity $v(0)$ at time $t=0$, we have $v(t) \to \underbrace{\frac{mg}{k}}_{\text{terminal velocity } v_t}$ as $t \to \infty$.

② Solution formula for linear equations

- standard form: $\frac{dv}{dt} = -\frac{k}{m} v + g$ hence $a(t) = -\frac{k}{m}$, $b(t) = g$
- integrating factor: $\mu(t) = e^{-\int^t a(s)\,ds} = e^{-\int^t \frac{-k}{m}\,ds} = e^{kt/m}$
- general solution:

$$v(t) = \frac{1}{\mu(t)}\left[c + \int^t \mu(s)\, b(s)\, ds\right]$$

$$= e^{-kt/m}\left[c + \int^t e^{ks/m} g\, ds\right]$$

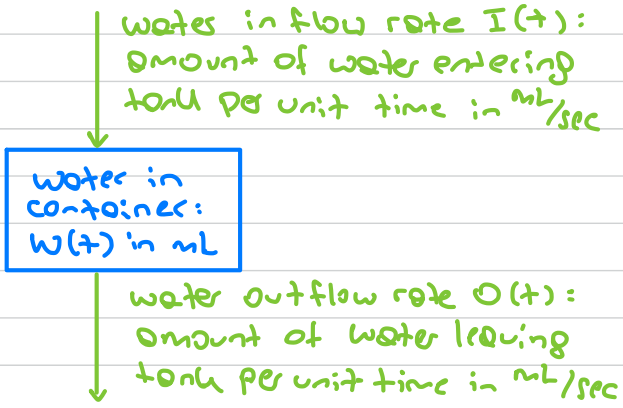
$$= e^{-kt/m}\left[c + \frac{mg}{k} e^{kt/m}\right]$$

$$\underbrace{v(t) = c e^{-kt/m} + \frac{mg}{k}}_{\text{general solution}} \to \frac{mg}{k} = v_t \quad \text{as } t \to \infty$$

(regardless of value of c)

→ Arrive at same conclusion as above!

Example 2 Compartment models : bathtub as example



Goal: describe the amount of water in the container at time t given the inflow and outflow rates

Pick a small time increment $h > 0$ (in sec), then

$$W(t+h) = W(t) + \text{inflow rate} \times \text{elapsed time} - \text{outflow rate} \times \text{elapsed time}$$
$$= W(t) + I(t)h - O(t)h$$

Hence, we find $$\frac{W(t+h) - W(t)}{h} = I(t) - O(t)$$

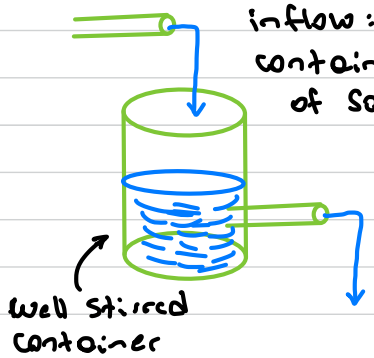
and taking the limit $h \to 0$ gives

$$\frac{dW}{dt} = I(t) - O(t)$$

balance equation

rate of change $\frac{dW}{dt}$ of water = amount entering per unit time − amount leaving per unit time

Example 3 Mixing



- time t counted in min
- at time $t = 0$, the tank contains Q_0 lb of salt dissolved in 500 gal of water

- Task : Find the amount of salt in the tank at any given time $t \geq 0$ and determine the limiting amount of salt as $t \to \infty$.

- Variables : $Q(t)$ = amount of salt in the tank at time t
  $W(t)$ = amount of water in the tank at time t

- Balance equation for salt $Q(t)$:

$$\frac{dQ}{dt}\ \frac{lb}{min} = \underbrace{\text{amount of salt entering the tank per unit time}}_{\gamma\ \frac{lb}{gal}\cdot 10\ \frac{gal}{min} = 10\gamma\ \frac{lb}{min}} - \underbrace{\text{amount of salt leaving the tank per unit time}}_{10\ \frac{gal}{min}\ \frac{Q(t)\ lb}{W(t)\ gal} = \frac{10\,Q(t)}{W(t)}\ \frac{lb}{min}}$$

Hence, the units agree, and we arrive at

$$\frac{dQ(t)}{dt} = 10\gamma - \frac{10\,Q(t)}{W(t)}$$

need to determine $W(t)$

- Balance equation for water $W(t)$:

The balance equation for $W(t)$ is given by

$$\frac{dW}{dt} = \underbrace{\text{inflow rate}}_{10\ \frac{gal}{min}} - \underbrace{\text{outflow rate}}_{10\ \frac{gal}{min}} = 0$$

Hence, $\int_0^t \frac{dW(t)}{dt}\,dt = \int_0^t 0\,dt \quad \Rightarrow \quad W(t) - W(0) = 0$

and we conclude $W(t) = W(0) = 500$.

- Return to balance equation for salt $Q(t)$:

$\longrightarrow$ The remaining equation for the amount $Q(t)$ of salt is

$$\frac{dQ}{dt} = 10\gamma - \frac{10\,Q}{500} = 10\gamma - \frac{Q}{50} \quad \text{with} \quad Q(0) = Q_0$$

$\longrightarrow$ Linear, autonomous, separable scalar first-order ODE

Use solution formula for linear equations:

- standard form: $\frac{dQ}{dt} = -\frac{1}{50}Q + 10\gamma$ $\qquad a(t) = -\frac{1}{50}$, $b(t) = 10\gamma$
- integrating factor: $\mu(t) = e^{-\int^t a(s)\,ds} = e^{t/50}$

- general solution:

$$Q(t) = \frac{1}{\mu(t)}\left[c + \int^t \mu(s)\, b(s)\, ds\right]$$

$$= e^{-t/50}\left[c + \int^t e^{s/50}\, 10\gamma\, ds\right]$$

$$= e^{-t/50}\left[c + 500\gamma\, e^{t/50}\right]$$

$$Q(t) = c\, e^{-t/50} + 500\gamma$$

- initial condition: $Q(0) = Q_0$ gives

$Q_0 = c + 500\gamma$ hence $c = Q_0 - 500\gamma$

→ $Q(t) = (Q_0 - 500\gamma)\, e^{-t/50} + 500\gamma$ is the amount of salt in the tank at any time t, and the limiting amount of salt as $t \to \infty$ is given by $\lim_{t\to\infty} Q(t) = 500\gamma$ (as expected).

§2.5 Existence and uniqueness (revisited)

Goals
- provide stronger version of the existence and uniqueness theorem
- discuss idea behind its proof

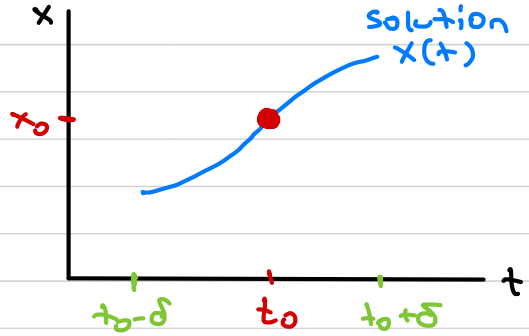
Consider the initial-value problem

(*) $\frac{dx}{dt} = f(x,t)$ with $x(t_0) = x_0$

Task: show that there is a (unique) continuously differentiable function $x(t)$ defined for all t near t_0 so that

$$\begin{cases} \frac{dx(t)}{dt} = f(x(t),t) \\ x(t_0) = x_0 \end{cases}$$

for all t near t_0.



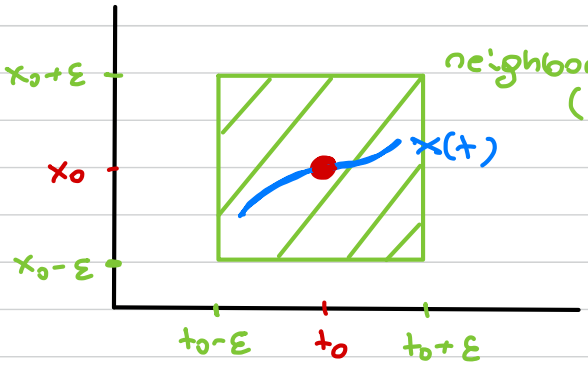
Existence and uniqueness theorem:

Fix x_0, t_0 and consider the initial-value problem

(*) $\frac{dx}{dt} = f(x,t)$ with $x(t_0) = x_0$.

(1) If there is an $\varepsilon > 0$ so that $f(x,t)$ is continuous in (x,t) for all x,t with $|x-x_0|, |t-t_0| < \varepsilon$, then there are a constant $\delta > 0$ and a continuously differentiable function $x(t)$ defined for $|t-t_0| < \delta$ so that $x(t)$ satisfies (*).

(2) If there is an $\varepsilon > 0$ so that $f(x,t)$ and $\frac{\partial f}{\partial x}(x,t)$ are continuous in (x,t) for all x,t with $|x-x_0|, |t-t_0| < \varepsilon$, then there are a constant $\delta > 0$ and a unique continuously differentiable function $x(t)$ defined for $|t-t_0| < \delta$ so that $x(t)$ satisfies (*).



neighborhood of (x_0,t_0) on which $f(x,t)$ (and $\frac{\partial f}{\partial x}(x,t)$) are well defined and continuous in (x,t).

Improvements of previous statement:

- need continuity of f and $\frac{\partial f}{\partial x}$ only near (x_0,t_0).
- separates existence from uniqueness:
  - f continuous implies existence
  - f and $\frac{\partial f}{\partial x}$ continuous implies also uniqueness

Idea of proof:

We want to find a function $x(t)$ so that

$$\begin{cases} \frac{dx(t)}{dt} = f(x(t),t) & \text{for all } t \text{ near } t_0 \\ x(t_0) = x_0 \end{cases}$$

Integrating the differential equation and using the fundamental theorem of calculus, we get

$$\int_{t_0}^{t} \frac{dx(s)}{ds}\, ds = \int_{t_0}^{t} f(x(s),s)\, ds \qquad \text{for all } t \text{ near } t_0$$

$$x(t) - \underbrace{x(t_0)}_{=x_0} = \int_{t_0}^{t} f(x(s),s)\, ds$$

„!" means „find a function $x(t)$ so that $x(t) =$ RHS for all t near t_0"

$$(**) \qquad x(t) \overset{!}{=} x_0 + \int_{t_0}^{t} f(x(s), s)\, ds \qquad \text{for all } t \text{ near } t_0$$

for each function $x(t)$, the right-hand side defines a new function $y(t)$ of t →
we need to show that there is a function $x(t)$ that reproduces itself when substituted into the right-hand side: this function is a „fixed point" under the transformation that brings us from $x(t)$ to $y(t)$.

→ • Brouwer's fixed-point theorem can be used to show existence of fixed points of (**) and therefore solutions of (*).

Analogy:



- Briefly stir a glass of water
- At least one water molecule will be back at its original position (this molecule is a fixed point)

• Banach's fixed-point theorem needs more assumptions but also guarantees uniqueness of fixed points of (**).

Analogy:



contraction F

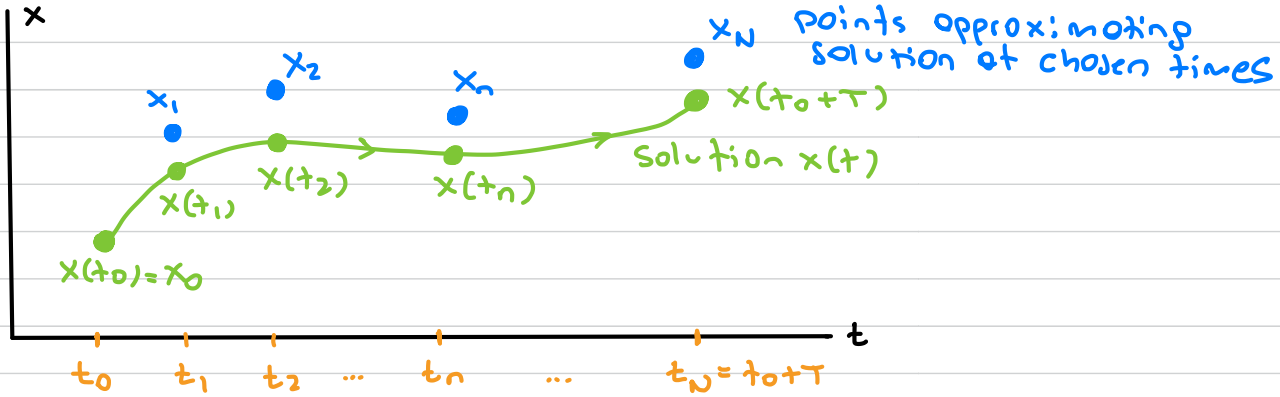


$|F(x) - F(y)| \leq \frac{1}{2}|x-y|$
so points move towards each other with rate strictly less than one

§2.6 Numerical methods

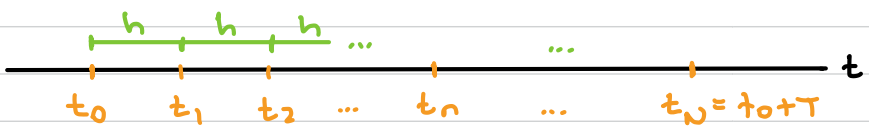
Goals
- How can we solve differential equations numerically on computers?
- Which algorithms might packages such as Matlab or Python (in our Jupyter notebooks) use?

Setting
- Consider (*) $\frac{dx}{dt} = f(x,t)$ with $x(t_0) = x_0$
- Assume that $f(x,t)$ satisfies the conditions of the existence & uniqueness theorem and let $x(t)$ denote the unique solution of (*) for $t_0 \le t \le t_0 + T$.



Given $N+1$ time points t_n for $0 \le n \le N$ so that
$t_0 < t_1 < t_2 < \dots < t_n < \dots < t_N = t_0 + T$,
find points x_n for $0 \le n \le N$ that approximate $x(t_n)$ so that $\max_{0 \le n \le N} |x(t_n) - x_n|$ is small.

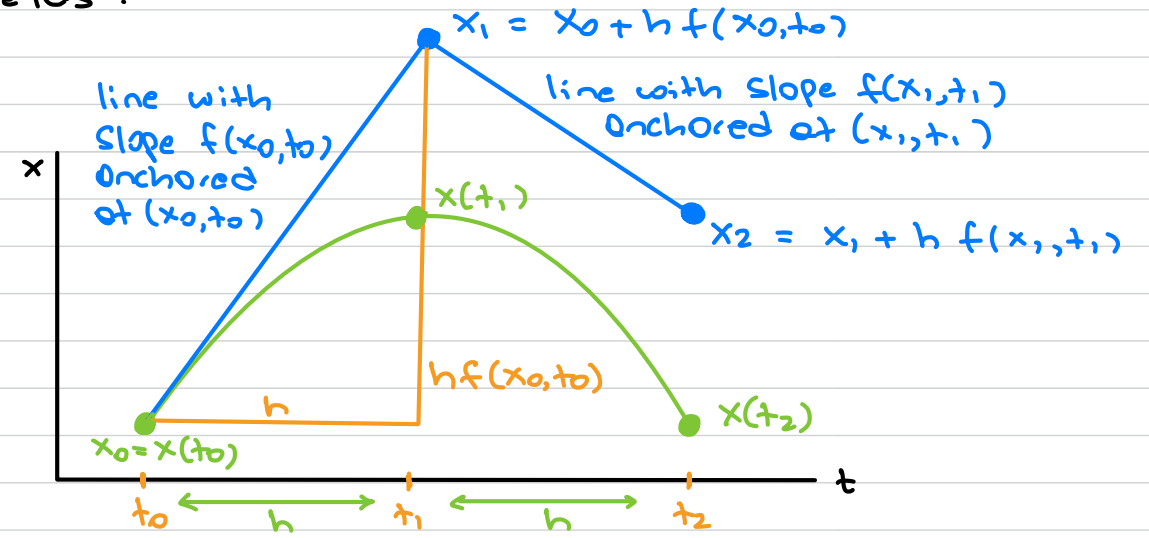
Approach choose equidistant points t_n between t_0 and t_0+T:
Given $N \gg 1$ large, we define $h := T/N \ll 1$ small and
$t_n = t_0 + nh$ for $0 \le n \le N$
so that $t_N = t_0 + Nh = t_0 + T$:



→ find approximations x_n of $x(t_n)$ <u>without</u> knowing the solution $x(t)$.

→ We have the following information:
- we know x_0 at $t = t_0$
- we can evaluate $f(x,t)$ at any given point (x,t).

Direction fields :



Algorithm in formulas for $t_n = t_0 + nh$:

(**)
- x_0 at $t = t_0$ given
- $x_1 = x_0 + h f(x_0,t_0)$
- $x_2 = x_1 + h f(x_1,t_1)$
- ...
- $x_{n+1} = x_n + h f(x_n,t_n)$
- ...
- $x_N = x_{N-1} + h f(x_{N-1},t_{N-1})$

each step uses only information from previous step plus evaluation of $f(x,t)$

If $h>0$ is small, we expect that x_n as determined by (**) is close to $x(t_n)$ for $0 \le n \le N$.

Forward Euler method :

Goal : approximate solution $x(t)$ of $\frac{dx}{dt} = f(x,t)$ with $x(t_0) = x_0$ for $t_0 \le t \le t_0 + T$

Algorithm :
1. Pick $N \gg 1$ large and set $h = \frac{T}{N}$
2. Recursively, define $x_{n+1} := x_n + h f(x_n, t_0 + nh)$ $\quad n = 0, \dots, N-1$

Theorem : There is a constant $C > 0$ so that $|x(t_0 + nh) - x_n| \le Ch$ for all $0 \le n \le N$

Alternative derivation: Balance equation or Taylor expansion

- We derived $\frac{dx}{dt} = f(x,t)$ from a balance equation:
  Pick $0 < h \ll 1$ small, then (elapsed time; inflows and outflows)
  $$x(t+h) \approx x(t) + h\, f(x(t),t)$$

- Alternatively, Taylor expansion shows that
  $$\left| x(t+h) - x(t) - h\,\frac{dx(t)}{dt} \right| \le C h^2 \quad \text{or}$$
  $$x(t+h) = x(t) + h\, f(x(t),t) + \underbrace{[\dots]}_{\le C h^2} \quad \text{because} \quad \frac{dx(t)}{dt} = f(x(t),t)$$

→ $t = t_0$: $x(t_0+h) = x(t_0) + h\, f(x(t_0), t_0) + [\dots]$ or

$$x(t_0+h) = x_0 + h\, f(x_0,t_0) + \underbrace{[\dots]}_{\le C h^2}$$

omit to obtain computable approximation

Define $x_1 := x_0 + h\, f(x_0,t_0)$ — unknown but much smaller than $h f(x_0,t_0)$

→ Can now define x_{n+1} recursively using x_n:

$$x(t_0+(n+1)h) = \underbrace{x(t_0+nh)}_{\text{approximated by } x_n} + h\, f(\underbrace{x(t_0+nh)}_{\text{approximated by } x_n}, t_0+nh) + \underbrace{[\dots]}_{\le C h^2}$$

Omit

Define $x_{n+1} := x_n + h\, f(x_n, t_0+nh)$: approximation of $x(t_0+(n+1)h)$

Linear algebra review

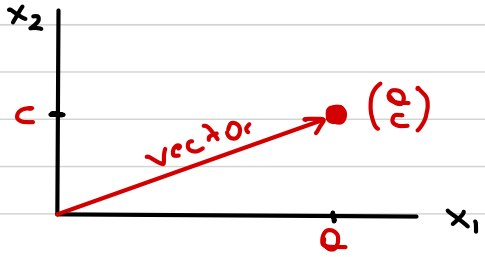
- Focus on two coupled equations that depend linearly on two variables.
- Throughout, the constants a, b, c, d are given, we may vary the right-hand sides g_1, g_2, and we want to solve for x_1, x_2.

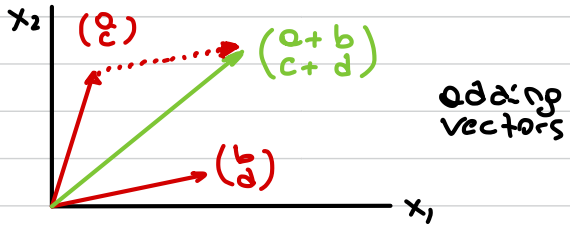
Systems of linear equations:

$$(*) \quad \begin{cases} a x_1 + b x_2 = g_1 \\ c x_1 + d x_2 = g_2 \end{cases} \quad \rightarrow \text{want to find } x_1, x_2$$

→ can solve 1st equation for x_1 as a function of x_2, then substitute this for x_1 into the second equation, and finally solve for x_2 (if possible)

Coordinates and vectors in the plane:





Matrix-vector notation:

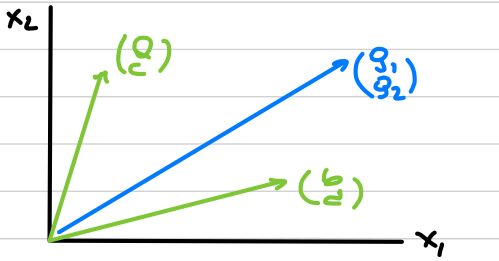
write (1) as

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} a x_1 + b x_2 \\ c x_1 + d x_2 \end{pmatrix} = \begin{pmatrix} a x_1 \\ c x_1 \end{pmatrix} + \begin{pmatrix} b x_2 \\ d x_2 \end{pmatrix}$$

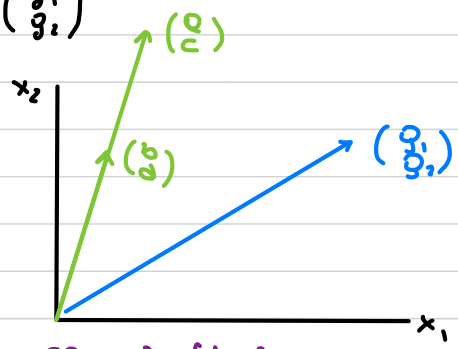
$$= \begin{pmatrix} a \\ c \end{pmatrix} x_1 + \begin{pmatrix} b \\ d \end{pmatrix} x_2 =: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solvability of (*):

Find x_1, x_2 so that $\begin{pmatrix} a \\ c \end{pmatrix} x_1 + \begin{pmatrix} b \\ d \end{pmatrix} x_2 = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$

① 

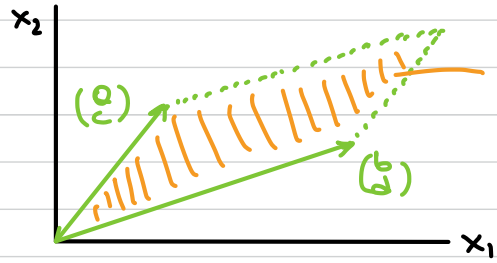
x_1, x_2 exist and are unique!

② 

cannot find x_1, x_2

→ no solution unless $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ is parallel to $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$

Distinguish cases ① and ② analytically:



area $= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left|\begin{matrix} a & b \\ c & d \end{matrix}\right| = ad - bc$

$\rightarrow$ area $= 0 \iff \binom{a}{c}, \binom{b}{d}$ are parallel

Conclusion: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

(1) $\det A = 0$ if and only if $A\binom{x_1}{x_2} = 0$ has a solution $\binom{x_1}{x_2} \neq 0$

(2) $\det A \neq 0$ if and only if for each $\binom{g_1}{g_2}$ the equation $A\binom{x_1}{x_2} = \binom{g_1}{g_2}$ has a unique solution $\binom{x_1}{x_2}$.

this solution is given by $\binom{x_1}{x_2} = A^{-1}\binom{g_1}{g_2}$

where $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Notation: $\binom{a}{c}$ and $\binom{b}{d}$ are said to be linearly independent if and only if $\left|\begin{matrix} a & b \\ c & d \end{matrix}\right| \neq 0$ (they are not parallel)

Examples.

- Find all solutions $\binom{x_1}{x_2}$ of $\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\binom{x_1}{x_2} = 0$

  Answer: $\left.\begin{matrix} x_1 - x_2 = 0 \\ -2x_1 + 2x_2 = 0 \end{matrix}\right\}$ hence, the only restriction is $x_1 = x_2$ and each solution is of the form $\binom{x_1}{x_2} = c\binom{1}{1}$ where $c \in \mathbb{R}$ is arbitrary.

- Find all solutions $\binom{x_1}{x_2}$ of $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\binom{x_1}{x_2} = \binom{2}{1}$

  Answer: $\left.\begin{matrix} x_1 = 2 \\ 2x_1 + x_2 = 1 \end{matrix}\right\}$ hence, $\binom{x_1}{x_2} = \binom{2}{-3}$ is the only solution

- Does $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\binom{x_1}{x_2} = \binom{g_1}{g_2}$ have a unique solution for each $\binom{g_1}{g_2}$?

  Answer: (2) states that the answer is yes if and only if $\left|\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}\right| \neq 0$.

  We have $\left|\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}\right| = 1 - (-1) = 2 \neq 0$, so the answer is yes ✓

- For which λ does $\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\binom{x_1}{x_2} = 0$ have a solution $\binom{x_1}{x_2} \neq 0$?

  Answer: (1) states that λ needs to satisfy $\left|\begin{matrix} -\lambda & 1 \\ 1 & -\lambda \end{matrix}\right| = 0$.

  Since $\left|\begin{matrix} -\lambda & 1 \\ 1 & -\lambda \end{matrix}\right| = \lambda^2 - 1$, this holds precisely when $\lambda = 1$ or $\lambda = -1$.

§3 Systems of differential equations

§3.1 Linear first-order systems in $\mathbb{R}^2$

We focus on ODEs of the form

$$\begin{cases} \dot{u} = au + bv \\ \dot{v} = cu + dv \end{cases} \quad \text{or} \quad \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where a, b, c, d are real constants and $\dot{u} = \frac{du}{dt}$

Example: We will find all solutions of

$$(*) \quad \begin{cases} \dot{u} = v \\ \dot{v} = u \end{cases} \quad \text{or} \quad \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a 2×2 matrix

- $\begin{pmatrix} u \\ v \end{pmatrix}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$ for all t is a solution of $(*)$.
  → rest state or equilibrium

- Find nonzero solutions of $(*)$:
  We seek solutions with exponential dependence on t

  Let $\begin{pmatrix} u \\ v \end{pmatrix}(t) = e^{\lambda t}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$
  (λ undetermined; $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ undetermined constant nonzero vector)

  Substitution into $(*)$ $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix}$ gives

  LHS $= \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{d}{dt}\left[e^{\lambda t}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\right] = \lambda e^{\lambda t}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$

  RHS $= A\begin{pmatrix} u \\ v \end{pmatrix} = A e^{\lambda t}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = e^{\lambda t} A\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$

  (does not use the specific form of A)

  LHS = RHS gives

  $$\lambda e^{\lambda t}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = e^{\lambda t} A\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \quad : \text{divide by } e^{\lambda t} \text{ which is always positive}$$

  $$\lambda\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = A\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

  $$\left[A - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = 0$$

  → need to find all λ for which $\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = 0$ has a nonzero solution $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Linear algebra review: $\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = 0$ has a nonzero solution $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \neq 0$ if and only if $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$

(Conclusion (1))

Hence, we need $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (-\lambda)^2 - 1^2 = \lambda^2 - 1 = 0$

which holds precisely when $\lambda = \pm 1$ ← we now know that only $\lambda = \pm 1$ can yield nonzero solutions $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \neq 0$ of $\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = 0$

→ Separately for $\lambda = 1$ and $\lambda = -1$, we need to find the associated nonzero vectors $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ for which $\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = 0$

$\lambda = 1$: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = 0$ gives $\left.\begin{aligned} -u_0 + v_0 &= 0 \\ u_0 - v_0 &= 0 \end{aligned}\right\}$ same equation as expected ✓

→ need to solve only one of them: $u_0 = v_0$

→ $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ where $c_1 \in \mathbb{R}$ is an arbitrary constant

$\lambda = -1$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = 0$ gives $\left.\begin{aligned} u_0 + v_0 &= 0 \\ u_0 + v_0 &= 0 \end{aligned}\right\}$ same equation again ✓

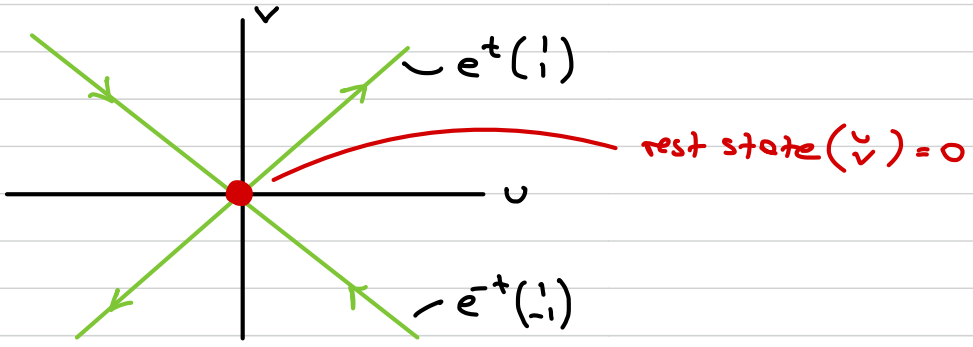
→ $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ where $c_2 \in \mathbb{R}$ is an arbitrary constant

- **Conclusion:** The functions $\begin{pmatrix} u \\ v \end{pmatrix}(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} u \\ v \end{pmatrix}(t) = c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are solutions of

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

for any choice of constants $c_1, c_2 \in \mathbb{R}$

Preliminary phase portrait



→ How can we find other solutions?

Superposition principle

Let $A \in \mathbb{R}^{n\times n}$. If $x_1(t)$ and $x_2(t)$ are two solutions of $\frac{dx}{dt} = Ax$ and $c_1, c_2 \in \mathbb{R}$ are constants, then $x(t) := c_1 x_1(t) + c_2 x_2(t)$ is also a solution of $\frac{dx}{dt} = Ax$.

Proof We know that $\frac{dx_1(t)}{dt} = Ax_1(t)$ and $\frac{dx_2}{dt}(t) = Ax_2(t)$ for all times t since $x_1(t)$ and $x_2(t)$ satisfy $\frac{dx}{dt} = Ax$.

We need to show that $x(t) = c_1x_1(t) + c_2x_2(t)$ is also a solution.

$$\text{LHS} = \frac{dx}{dt} = \frac{d}{dt}(c_1x_1(t) + c_2x_2(t))$$

$$= c_1\frac{dx_1(t)}{dt} + c_2\frac{dx_2(t)}{dt}$$ because c_1, c_2 are constants

$$= c_1Ax_1(t) + c_2Ax_2(t)$$ because $x_1(t), x_2(t)$ are solutions

$$\text{RHS} = Ax = A(c_1x_1(t) + c_2x_2(t))$$

$$= c_1Ax_1(t) + c_2Ax_2(t)$$ because of distributive law and c_1, c_2 are scalars

$\Rightarrow$ LHS = RHS for all times

$\Rightarrow$ $x(t) = c_1x_1(t) + c_2x_2(t)$ satisfies $\frac{dx}{dt} = Ax$ □

Return to Example:

Recall that $\begin{pmatrix} u \\ v \end{pmatrix}(t) = e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} u \\ v \end{pmatrix}(t) = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ both satisfy $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$. The superposition principle therefore guaranteer that

$$(*) \quad \begin{pmatrix} u \\ v \end{pmatrix}(t) = c_1e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is also a solution to $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

→ $(*)$ is the general solution of $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$:
For each initial condition $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$, there are constants c_1, c_2 so that $\begin{pmatrix} u \\ v \end{pmatrix}(t)$ given by $(*)$ satisfies $\begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$.

We will prove this claim in two different (equivalent) ways:

Approach 1: Find c_1, c_2 so that $\begin{pmatrix} u \\ v \end{pmatrix}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \overset{!}{=} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$

→ need to find c_1, c_2 so that $\begin{cases} c_1 + c_2 = u_0 \\ c_1 - c_2 = v_0 \end{cases}$

→ hence $c_1 = \frac{1}{2}(u_0 + v_0)$ and $c_2 = \frac{1}{2}(u_0 - v_0)$

→ $\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix}(t) = \frac{1}{2}(u_0+v_0)e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}(u_0-v_0)e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \text{satisfies } \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \text{ and } \begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \end{cases}$ ✓

Approach 2: Find c_1, c_2 so that $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$

→ has a unique solution $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ if and only if $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \neq 0$.

→ check: $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0$ ✓

In this case, we can find $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ by inverting the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} u_0 + v_0 \\ u_0 - v_0 \end{pmatrix}$$ ✓

Note: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ satisfies

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

recall $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

Conclusion of example 1 :

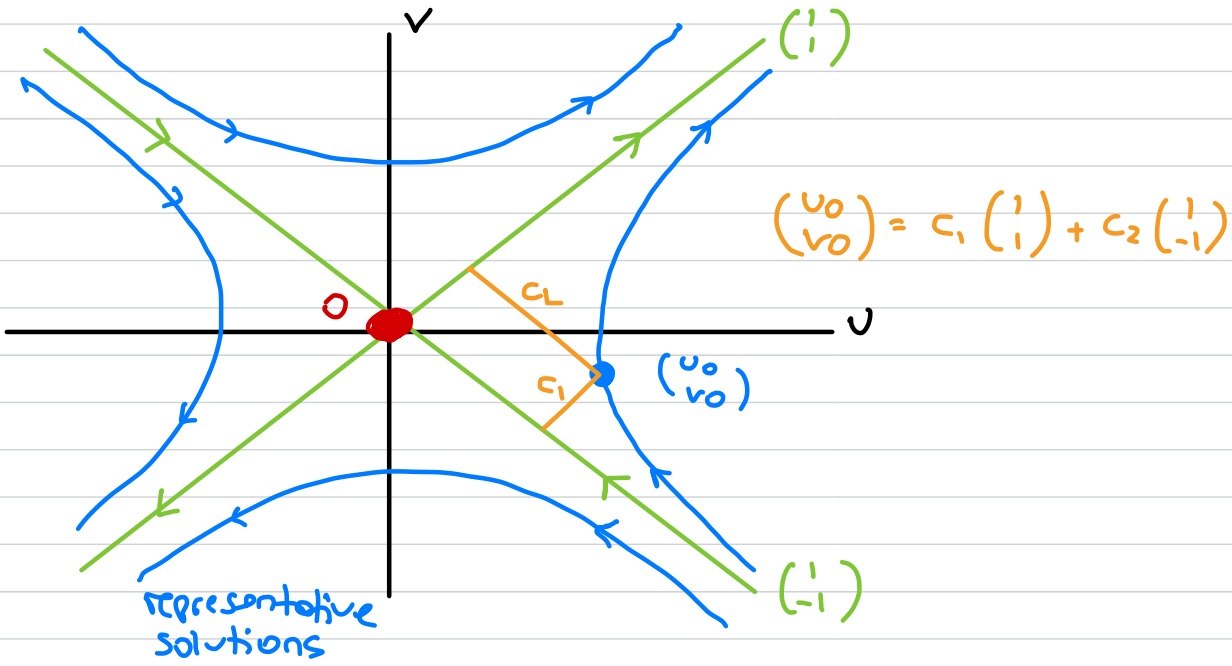
- The general solution of $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ is given by

  $\begin{pmatrix} u \\ v \end{pmatrix}(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ where c_1, c_2 are constants

- If we add the initial condition $\begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$, then

  $\begin{pmatrix} u \\ v \end{pmatrix}(t) = \frac{1}{2}(u_0+v_0) e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}(u_0-v_0) e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

  satisfies the ODE and the initial condition.



Summary: Consider $\dot{x} = Ax$ with $x \in \mathbb{R}^2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$

① Find roots λ_1, λ_2 of $\det\left[A - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \overset{!}{=} 0$
→ λ_1, λ_2 are the eigenvalues of A

② Find nonzero solutions $x_1, x_2 \in \mathbb{R}^2$ of $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$.
→ x_1, x_2 are the eigenvectors of A belonging to λ_1, λ_2.

③ Then $x(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$ is a solution of $\dot{x} = Ax$ where c_1, c_2 are arbitrary constants

④
- $x(t)$ is the general solution (meaning that for each given $x_0 \in \mathbb{R}^2$ there are constants c_1, c_2 so that $x(0) = x_0$) if and only if x_1, x_2 are linearly independent;
- The eigenvectors x_1, x_2 are automatically linearly independent when $\lambda_1 \neq \lambda_2$

Justification for ① - ③:

Approach 1: follow the steps in example 1 for the general case

Approach 2: Check LHS = RHS:

$$\text{LHS} = \frac{dx}{dt} = \frac{d}{dt}\left(c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2\right) = \lambda_1 c_1 e^{\lambda_1 t} x_1 + \lambda_2 c_2 e^{\lambda_2 t} x_2$$

$$\text{RHS} = Ax = c_1 e^{\lambda_1 t} \underbrace{Ax_1}_{=\lambda_1 x_1} + c_2 e^{\lambda_2 t} \underbrace{Ax_2}_{=\lambda_2 x_2} = c_1 e^{\lambda_1 t} \lambda_1 x_1 + c_2 e^{\lambda_2 t} \lambda_2 x_2$$

one LHS = RHS ✓

Justification for ④:

- We can solve $c_1 x_1 + c_2 x_2 = x_0$ for c_1, c_2 for each given initial condition if and only if x_1, x_2 are linearly independent (that is if and only if $\det(x_1, x_2) \neq 0$)
- If $\lambda_1 \neq \lambda_2$, then x_1, x_2 are linearly independent
→ shown in Linear Algebra

$x = \begin{pmatrix} v \\ w \end{pmatrix}$, $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Example 2: Solve $\dot{x} = \begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix} x$

① Find eigenvalues λ_1, λ_2 : $\det(A - \lambda 1) \stackrel{!}{=} 0$

$$\det\left[\begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{vmatrix} -2-\lambda & 3 \\ 0 & -5-\lambda \end{vmatrix} = (2+\lambda)(5+\lambda) \stackrel{!}{=} 0$$

Hence, $\lambda_1 = -2$, $\lambda_2 = -5$ (or vice versa: there is no natural numbering scheme)

② Find associated nonzero eigenvectors x_1, x_2 : $Ax_j = \lambda_j x_j$

$\lambda_1 = -2$:
$$\begin{pmatrix} -2-(-2) & 3 \\ 0 & -5-(-2) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 3w \\ -3w \end{pmatrix} \stackrel{!}{=} 0$$

Hence, $w = 0$. We pick $v = 1$ so that $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
(picking any nonzero value for v works)

$\lambda_2 = -5$
$$\begin{pmatrix} -2-(-5) & 3 \\ 0 & -5-(-5) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 3(v+w) \\ 0 \end{pmatrix} \stackrel{!}{=} 0$$

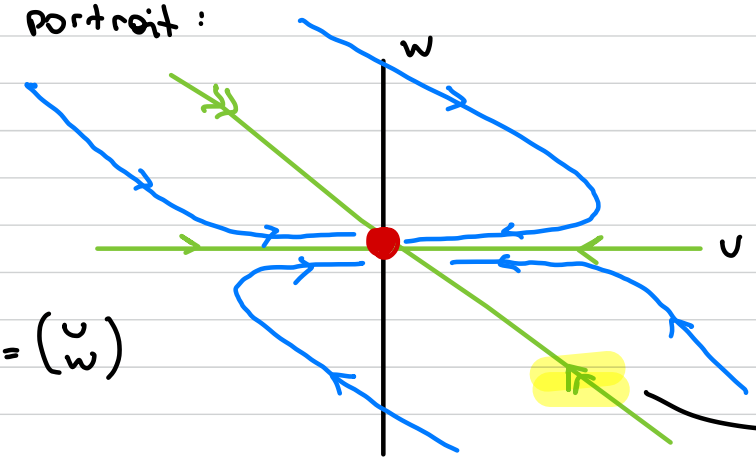
Hence, $w = -v$. We pick $v = 1$ so that $x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
(setting $v = -w$ and picking any nonzero value for w is equally valid.)

③ The function $x(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = c_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
is a solution to $\dot{x} = \begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix} x$.

④ Since $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent, $x(t)$ given in ③ is the general solution.

(two arguments:
1. $\lambda_1 \neq \lambda_2$
2. $\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1 \neq 0$)

⑤ Phase portrait:



all solutions converge to zero as $t \to \infty$

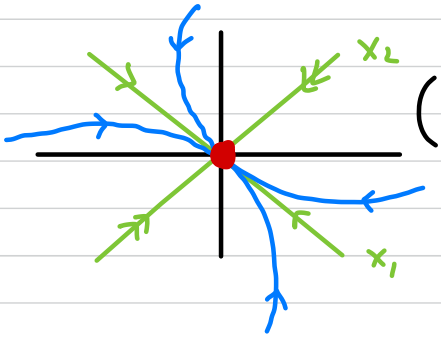
"$>>$" means solutions "decay stronger than" "$>$"

Classification of phase portraits for $\dot{x} = Ax$

Assume λ_1, λ_2 are the eigenvalues of A with associated eigenvectors x_1, x_2 and general solution $x(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

- **Stable** ($x=0$ is an attractor)

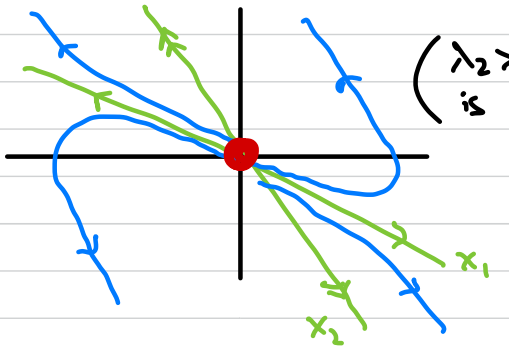
  If $\lambda_1, \lambda_2 < 0$,
  then $|x(t)| \to 0$ as $t \to \infty$



($\lambda_2 < \lambda_1 < 0$ is shown)

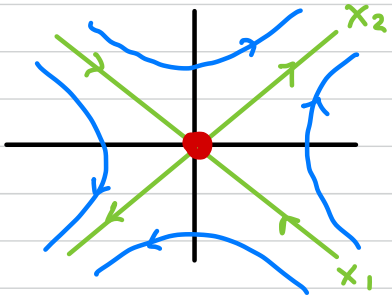
- **unstable** ($x=0$ is a repellor)

  If $\lambda_1, \lambda_2 > 0$,
  then $|x(t)| \to \infty$ as $t \to \infty$
  for all nonzero solutions

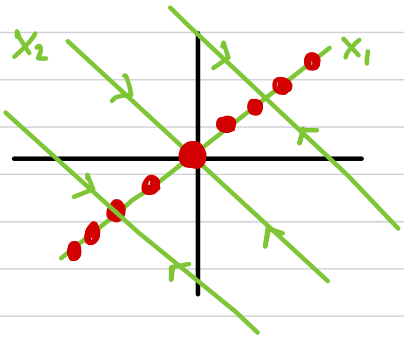


($\lambda_2 > \lambda_1 > 0$ is shown)

- **saddle**: $\lambda_1 < 0 < \lambda_2$



- **neutral**: $\lambda_1 = 0$ (or $\lambda_2 = 0$)



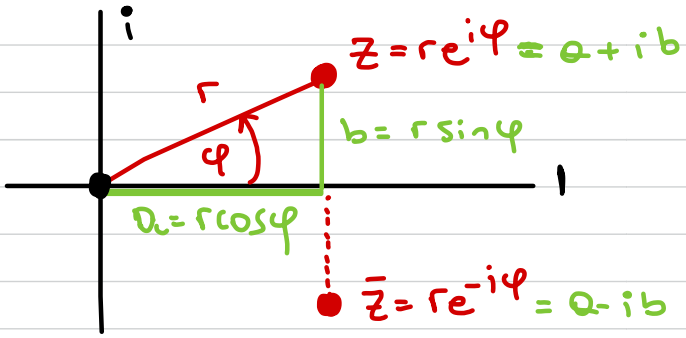
($\lambda_2 < 0$ shown)

Focus next on matrices with complex (non-real) eigenvalues

Primer on complex numbers:

- imaginary unit: $i := \sqrt{-1}$ so that $i^2 = -1$
- complex numbers: $z = a + ib$ with $a, b \in \mathbb{R}$
- real/imaginary parts: $z = a + ib \Rightarrow \begin{cases} a = \operatorname{Re} z & \text{real part} \\ b = \operatorname{Im} z & \text{imaginary part} \end{cases}$
- operations – add and multiply complex numbers ($i^2 = -1$)
  - complex conjugate: $z = a + ib \Rightarrow \bar{z} := a - ib$
- Euler's formula: $z = a + ib \Rightarrow e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$ well defined since we can add/multiply

  $\Rightarrow e^{i\varphi} = \cos\varphi + i\sin\varphi$ } use Taylor series for $e^x, \cos x, \sin x$
- geometry:



- norms/absolute value: $z = re^{i\varphi} = a + ib \Rightarrow |z| = |r| = \sqrt{a^2 + b^2}$
  → „length" of complex number

Example: Solve $\frac{dx}{dt} = Ax = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$

- eigenvalues:

$$|A - \lambda 1| = \left| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$\Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm\sqrt{-1} = \pm i$ hence $\lambda_{1,2} = \pm i$

- eigenvectors:

$\lambda_1 = i$ $\quad \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}\Big|_{\lambda=i} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -iv+w \\ -v-iw \end{pmatrix} = 0$

note: $i(-v-iw) = -iv - i^2 w = -iv + w$
($i(-v-iw)$: 2nd eqn. $\quad -iv+w$: 1st eqn.)
→ suffices to solve one of these two equations

Solve $-iv+w=0 \Rightarrow w = iv$
pick $v=1$ so that $w=i \Rightarrow x_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

→ $x(t) = e^{\lambda_1 t} x_1 = e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} = (\cos t + i \sin t)\begin{pmatrix} 1 \\ i \end{pmatrix}$ (Euler's formula)

$= \begin{pmatrix} \cos t + i \sin t \\ i\cos t - \sin t \end{pmatrix} = \underbrace{\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}}_{= \operatorname{Re} x(t) \,=: x_1(t)} + i \underbrace{\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}}_{= \operatorname{Im} x(t) \,=: x_2(t)}$

is a solution of $\dot{x} = Ax$.

Lemma If A has only real entries, and $x(t)$ is a complex-valued solution of $\dot{x} = Ax$, then $x_1(t) := \operatorname{Re} x(t)$ and $x_2(t) := \operatorname{Im} x(t)$ are also solutions of $\dot{x} = Ax$.

Proof We know that $\frac{dx(t)}{dt} = Ax(t)$ for all t.
Let $x(t) = x_1(t) + i x_2(t)$ where $x_1(t), x_2(t)$ are both real.

$\frac{dx(t)}{dt} = \frac{d}{dt}(x_1(t) + i x_2(t)) = \frac{dx_1(t)}{dt} + i \frac{dx_2(t)}{dt}$

$x(t)$ is a solution → $Ax(t) = A(x_1(t) + i x_2(t)) = Ax_1(t) + i Ax_2(t)$

Since A has only real entries, we conclude that $\frac{dx_1}{dt} = Ax_1$ and $\frac{dx_2}{dt} = Ax_2$ □

→ $x(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ is a solution of $\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$ for all values of c_1, c_2. (*)

linearly independent!

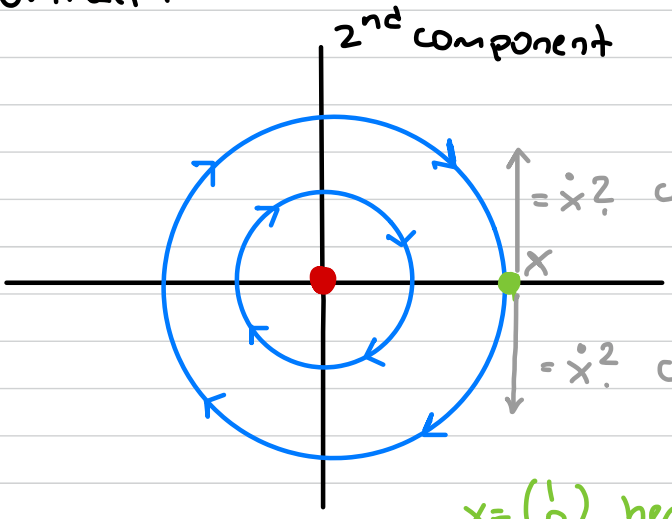
Note that $x(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

hence

$$x(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

is the general solution of $\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$

- phase portrait:



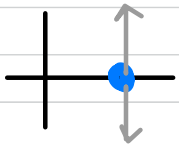
$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ hence $\dot{x} = Ax = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

→ clockwise rotation

General Process: $\dot{x} = Ax$

- eigenvalues: $|A - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}| = 0$ gives $\lambda = \lambda_{1,2}$
  focus here on $\lambda_{1,2} = a \pm ib$ with $b \neq 0$
- eigenvector for $\lambda = a + ib$:

  find $x_1 \neq 0$ so that $\left(A - \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) x_1 = 0$
- Solution $\tilde{x}(t) = e^{\lambda_1 t} x_1 = e^{at} (\cos bt + i \sin bt)\, x_1$
- general solution: $x(t) = c_1 \operatorname{Re} \tilde{x}(t) + c_2 \operatorname{Im} \tilde{x}(t)$
  (it can be shown that this always gives the general solution)
- phase portrait:

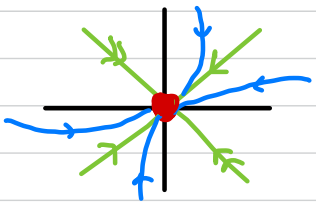


use $\dot{x} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to determine whether solution rotates clockwise or counterclockwise

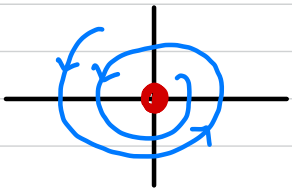
Classification (revisited):

Let $\dot{x} = f(x)$ with $x \in \mathbb{R}^2$
Let equilibrium x_* (so that $f(x_*) = 0$) and Jacobian $A := \frac{df}{dx}(x_*)$
Assume λ_1, λ_2 are the eigenvalues of A

- x_* is called stable, or an attractor, if $\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2 < 0$
  $\Rightarrow$ $x(t) \to x_*$ as $t \to \infty$ for all $x(0)$ near x_*



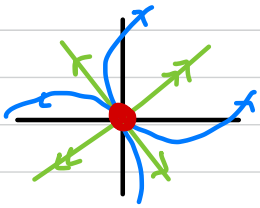
stable node
λ_1, λ_2 real



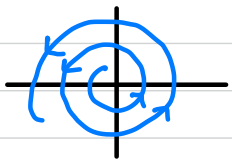
stable spiral
λ_1, λ_2 not real

phase portraits for $\dot{x} = Ax$ and $\dot{x} = f(x)$ near x_* look very similar

- x_* is called repelling (or unstable), or a repeller, if $\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2 > 0$
  $\Rightarrow$ $x(t) \to x_*$ as $t \to -\infty$ for all $x(0)$ near x_*

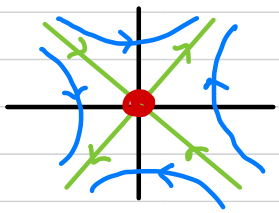


unstable node
λ_1, λ_2 real

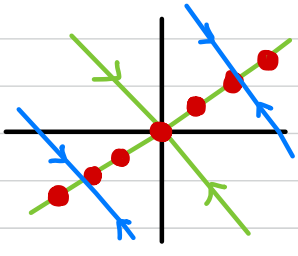


unstable spiral
λ_1, λ_2 not real

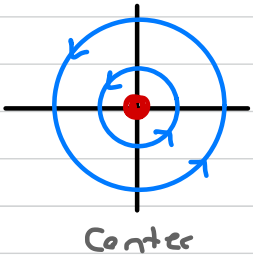
- x_* is called a saddle if $\lambda_1 < 0 < \lambda_2$



- x_* is called neutral if $\lambda_1 = 0$
  -"- center if $\lambda_{1,2} = \pm i\omega$ with $\omega \neq 0$



neutral



center

- phase portraits for $\dot{x} = Ax$ are shown;
- phase portraits for $\dot{x} = f(x)$ near x_* may be different

Summary: considered $\frac{dx}{dt} = Ax = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} x$ with $x = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$

→ eigenvalues $\lambda_{1,2} = 3 \pm 2i$
eigenvector belonging to $\lambda_1 = 3+2i$ is $x_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ (or any nonzero multiple)

→ $x(t) = e^{\lambda_1 t} x_1 = e^{(3+2i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} = e^{3t}(\cos 2t + i \sin 2t) \begin{pmatrix} i \\ 1 \end{pmatrix}$

$$x(t) = \underbrace{e^{3t} \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}}_{= \operatorname{Re} x(t) =: x_1(t)} + i \underbrace{e^{3t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}}_{= \operatorname{Im} x(t) = x_2(t)}$$

is a complex-valued solution of $\dot{x} = Ax$

Lemma: If all entries in A are real, and $x(t)$ is a complex-valued solution of $\dot{x} = Ax$, then $x_1(t) := \operatorname{Re} x(t)$ and $x_2(t) := \operatorname{Im} x(t)$ are (real-valued) solutions of $\dot{x} = Ax$.

Proof: We know that $\dot{x} = Ax$ for all t. Let $x(t) = x_1(t) + i x_2(t)$ where $x_1(t), x_2(t)$ are both real-valued:

$$\frac{dx(t)}{dt} = \frac{d}{dt}(x_1(t) + i x_2(t)) = \boxed{\frac{dx_1(t)}{dt}} + i \boxed{\frac{dx_2(t)}{dt}}$$

because $x(t)$ is a solution → $=$

$$Ax(t) = A(x_1(t) + i x_2(t)) = \boxed{Ax_1(t)} + i \boxed{Ax_2(t)}$$

real numbers / real numbers

Since A has only real entries, we conclude that $\dot{x}_1 = Ax_1$ and $\dot{x}_2 = Ax_2$ for all t. □

→ $x(t) = c_1 e^{3t} \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$ is a solution of $\dot{x} = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} x$ for each choice of the constants c_1, c_2.

→ Since $x(0) = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (linearly independent!), we conclude the following:

$$\boxed{\begin{array}{l} x(t) = c_1 e^{3t} \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} \\ \text{is the general solution of } \dot{x} = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} x \end{array}}$$

In particular, for non-real complex exponents, we only need to compute one of the two eigenvectors.

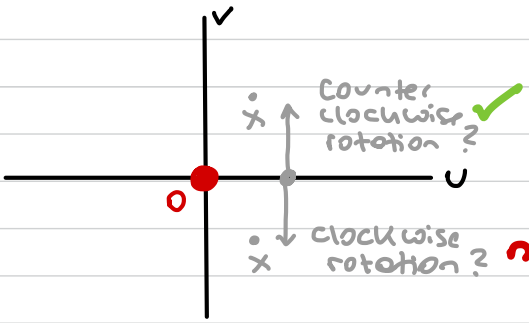
Phase portrait:

$$\dot{x} = Ax = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} x \quad \text{with general solution}$$

$$x(t) = c_1 e^{3t}\begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$$

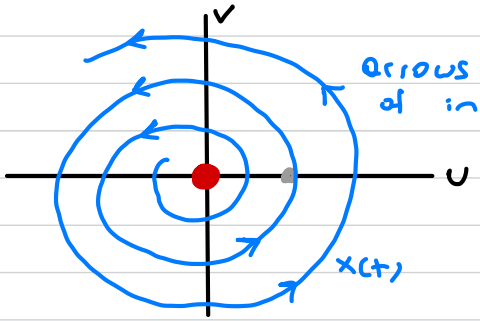
exponential growth as t increases

rotations / oscillations



$$\dot{x} = Ax\big|_{x=\binom{1}{0}} = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

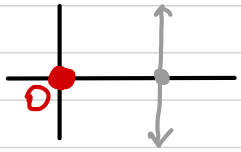
: points into upper half-plane → counter-clockwise rotation



arrows indicate direction of increasing time t.

General process: $\dot{x} = Ax = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x$

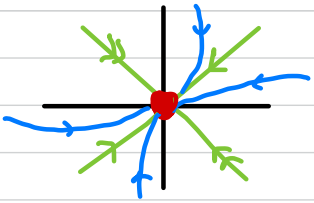
- eigenvalues: $\left| A - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$ gives $\lambda = \lambda_{1,2}$
  focus on $\lambda_{1,2} = r \pm i\omega$ with $\omega \neq 0$.
- eigenvector: find $x_1 \neq 0$ so that $\left[A - \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] x_1 = 0$
- solution: $\tilde{x}(t) = e^{\lambda_1 t} x_1 = e^{rt}(\cos \omega t + i \sin \omega t)\, x_1$
- general solution: $x(t) = c_1 \operatorname{Re} \tilde{x}(t) + c_2 \operatorname{Im} \tilde{x}(t)$
  (can show that this always gives the general solution!)
- phase portrait:
  - $r < 0$ stable ($x(t) \to 0$ as $t \to \infty$)
  - $r > 0$ unstable ($|x(t)| \to \infty$ as $t \to \infty$)



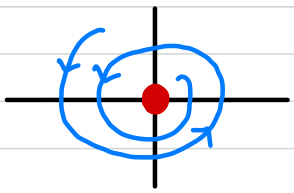
use $\dot{x} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ to determine whether solutions rotate clockwise or counterclockwise

Classification: $\dot{x} = Ax$ where A has eigenvalues λ_1, λ_2

- $\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2 < 0$ with $\lambda_1 \neq \lambda_2$ so that $x(t) \to 0$ as $t \to \infty$

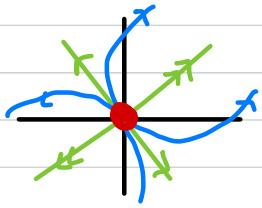


stable node
(λ_1, λ_2 real)

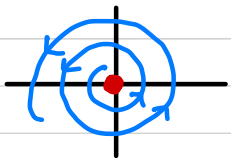


stable spiral
(λ_1, λ_2 not real)

- $\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2 > 0$ with $\lambda_1 \neq \lambda_2$ so that $|x(t)| \to \infty$ as $t \to \infty$

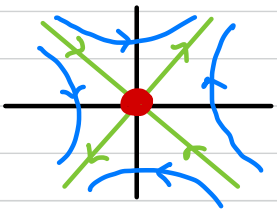


unstable node
(λ_1, λ_2 real)



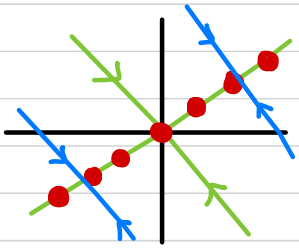
unstable spiral
(λ_1, λ_2 not real)

- $\operatorname{Re}\lambda_1 < 0 < \operatorname{Re}\lambda_2$

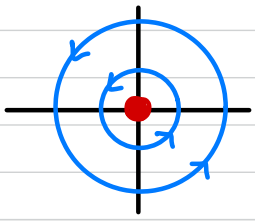


saddle

- $\begin{cases} \lambda_1 = 0 & : \text{ neutral} \\ \lambda_{1,2} = \pm i\omega \text{ for } \omega \neq 0 & : \text{ center} \end{cases}$



neutral
($\lambda_2 < 0$)



center

§3.2 Matrix exponentials

Motivation:

① Consider $\frac{dx}{dt} = Ax$. We found solutions in the form $x(t) = e^{\lambda t}\begin{pmatrix} v_0 \\ v_0 \end{pmatrix}$

This gave the general solution via the superposition principle when

① λ_1, λ_2 real with (i) $\lambda_1 \neq \lambda_2$

(ii) $\lambda_1 = \lambda_2$ with two linearly independent eigenvectors

② λ_1, λ_2 not real

→ have not dealt with the case $\lambda_1 = \lambda_2$ with only one eigenvector:

Example: $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ has $\lambda_1 = \lambda_2 = a$ with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

② Generalize to larger systems of the form $\frac{dx}{dt} = Ax$ where A $n \times n$ matrix and $x \in \mathbb{R}^n$

→ We have exploited that $\dot{x} = ax$ for $x \in \mathbb{R}$ has the solution $x(t) = e^{at} x_0$

→ By analogy, the solution to $\dot{x} = Ax$ where A is now an $n \times n$ matrix ought to be $x(t) = e^{At} x_0$

(e^{At}: $n \times n$ matrix; x_0: vector in $\mathbb{R}^n$)

→ what is „e^{At}"?

Recall: $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{k!}x^k + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$

(makes sense also for matrices!)

Definition: If A is an $n \times n$ matrix, we define

$$e^A := 1 + A + \frac{1}{2!}A^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

where $1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ is the $n \times n$ identity matrix

Caution ① e^A is not the same as exponentiating each entry in A

② Generally, $e^{A+B} \neq e^A e^B$ (see later for examples)

Lemma: The solution $x(t)$ of $\dot{x} = Ax$ with $x(0) = x_0$ is $x(t) = e^{At} x_0$

Proof: (*) $e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = 1 + At + \dots + \frac{t^k}{k!} A^k + \dots$

- Initial condition: $x(0) = e^{A \cdot 0} x_0 \overset{(*)}{=} 1 \cdot x_0 = x_0$
- LHS = RHS:

$$\frac{d}{dt} e^{At} = \frac{d}{dt} \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \frac{d}{dt} \left(1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} A^k \right) = \frac{d}{dt} \sum_{k=1}^{\infty} \frac{t^k}{k!} A^k$$

LHS

$$\overset{(**)}{=} \sum_{k=1}^{\infty} \frac{k t^{k-1}}{k!} A^k = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^k = A \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^{k-1}$$

$$= A \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \underbrace{A e^{At}}_{\text{RHS}}$$

(**) we can show that differentiation and summation commute for this series $\square$

$\longrightarrow$ How can we calculate e^{At} ?

Linear algebra: Let A be a real $n \times n$ matrix

- $\lambda \in \mathbb{C}$ is said to be an eigenvalue of A with eigenvector $v \neq 0$ if $Av = \lambda v$
- Each eigenvalue λ is a root of the nth-order polynomial $\det(A - \lambda 1) = 0$
- Each matrix A has n eigenvalues $\lambda_1, \dots, \lambda_n$ with eigenvectors $v_1, \dots, v_n$.
- If the eigenvalues are pairwise distinct, the eigenvectors $v_1, \dots, v_n$ satisfy $\det V \neq 0$ where $V = (v_1 | \dots | v_n)$ is the $n \times n$ matrix with columns given by $v_1, \dots, v_n$.

Note: If the eigenvalues $\lambda_1, \dots, \lambda_n$ are not pairwise distinct (that is, we have $\lambda_i = \lambda_j$ for some $i \neq j$), then we may or may not be able to find n eigenvectors $v_1, \dots, v_n$ for which $\det V \neq 0$ where $V = (v_1 | \dots | v_n)$.

$\longrightarrow$ See next example!

Example: repeated eigenvalues ($\lambda_1 = \lambda_2$)

① $\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x$

- eigenvalues and eigenvectors:

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \quad \text{hence } \lambda_1 = \lambda_2 = 1$$

$$\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \overset{\lambda=1}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \quad \text{regardless of values of } u, v$$

$\Rightarrow$ every vector is an eigenvector!

$\Rightarrow$ choose (for instance) $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

we want x_1, x_2 to be linearly independent!

- general solution: $x(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^t \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

② $\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$

- eigenvalues: $\left| \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0$

hence again $\lambda_1 = \lambda_2 = 0$

- eigenvectors: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix} = 0$ requires $v = 0$.

hence $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the only eigenvector

- solution: $x(t) = c_1 e^{\lambda_1 t} x_1 = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ← only one constant!

→ not clear how we can find the general solution in this case!

Conclusion

- In example ①, there are two eigenvectors v_1, v_2 of A with $\det(v_1 | v_2) \neq 0$, and we can solve the ODE using our previous approach
- In example ②, A has only one eigenvector, and our previous approach does not allow us to find the general solution.

Lemma: Assume $A \in \mathbb{R}^{n\times n}$ has eigenvalues $\lambda_1, \dots, \lambda_n$ and that the associated eigenvectors $v_1, \dots, v_n$ satisfy $\det V \neq 0$ where $V = (v_1 | \dots | v_n)$ has columns $v_1, \dots, v_n$. Let $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ be the diagonal matrix with the eigenvalues $\lambda_1, \dots, \lambda_n$ on the diagonal. Then:

① $A = VDV^{-1}$

② $e^{At} = V e^{Dt} V^{-1} = V \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} V^{-1}$

Proof:

① Let $e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← entry 1 in jth row for $j = 1, \dots, n$, then

$Ve_j = (v_1 | \dots | v_n) \cdot e_j = v_j$ and therefore $V^{-1} v_j = e_j$ for $j = 1, \dots, n$. Hence

$VDV^{-1} v_j = VDe_j = V \lambda_j e_j = \lambda_j v_j = A v_j$ for all j

which implies $VDV^{-1} = A$.

② Note: $(VDV^{-1})^2 = VDV^{-1} \cdot VDV^{-1} = VD^2V^{-1}$ and therefore

$$(VDV^{-1})^k = \underbrace{VDV^{-1} \cdot VDV^{-1} \cdot \dots \cdot VDV^{-1}}_{k \text{ factors}} = VD^kV^{-1} \quad (*)$$

$$e^{At} = e^{VDV^{-1}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} (VDV^{-1})^k \overset{(*)}{=} \sum_{k=0}^{\infty} \frac{t^k}{k!} V D^k V^{-1}$$

$$= V \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} D^k \right) V^{-1} = V e^{Dt} V^{-1} = V \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} V^{-1} \quad \square$$

Example 1: Find the general solution of $\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$.

- eigenvalues $\lambda_1 = i$ with $v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\lambda_2 = \overline{\lambda_1} = -i$ with $v_2 = \overline{v_1} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.
- $V = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$, $D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $V^{-1} \overset{(*)}{=} \frac{1}{-2i} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}$

$$\rightarrow e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t} = V e^{Dt} V^{-1} = \frac{1}{2i} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} = \dots = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

simplify using $e^{it} = \cos t + i \sin t$

$$\rightarrow x(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} x_0 \quad \checkmark$$

(*) Recall: if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Example 2: Repeated eigenvalues (with only one eigenvector)

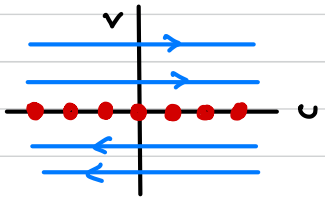
① Find the general solution of $\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$

→ use matrix exponentials:

- $x(t) = e^{At} x_0 = \left(1 + tA + \frac{t^2}{2!} A^2 + \dots + \frac{t^k}{k!} A^k + \dots\right) x_0$
- $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow A^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for all $k \geq 2$
- $x(t) = e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t} x_0 = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] x_0 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x_0$

→ the general solution of $\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$ with $x(0) = x_0$ is $x(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x_0$

phase portrait



$x_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ gives $x(t) = \begin{pmatrix} u_0 + t v_0 \\ v_0 \end{pmatrix}$

Lemma:
- If $AB = BA$, then $e^{A+B} = e^A e^B = e^B e^A$
- Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then $e^{A+B} \neq e^A e^B$

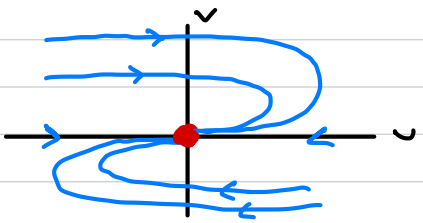
② More generally, the solution to $\dot{x} = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ with $x(0) = x_0$ is

(*) $x(t) = e^{at}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x_0$

Verify (*): $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} = a\,1 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $1 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot 1$

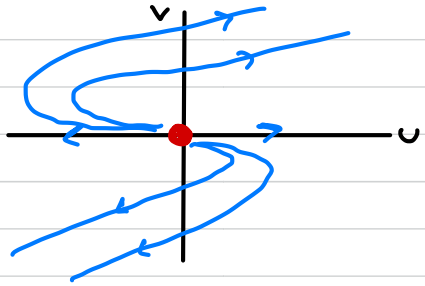
$\Rightarrow e^{At} = e^{\left(a1 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)t} = e^{a1t}\, e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t} = e^{at}\, 1 \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e^{at}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ ✓

Lemma above!



$\lambda = a < 0$ repeated

„stable improper node"



$\lambda = a > 0$ repeated

„unstable improper node"

§3.3 Inhomogeneous Systems

We now consider inhomogeneous systems of the form

$$(*) \qquad \frac{dx}{dt} = Ax + b(t) \qquad x \in \mathbb{R}^n$$

where A is an $n \times n$ matrix and $b(t)$ is a given continuous function with values in $\mathbb{R}^n$.

Lemma: The general solution $x(t)$ of $(*)$ is given by

$$x(t) = \underbrace{e^{At} x_0}_{\text{general solution of } \dot{x} = Ax} + \underbrace{\int_0^t e^{A(t-s)} b(s)\, ds}_{\text{particular solution of } \dot{x} = Ax + b(t)}$$

Proof: We can justify our statement in two ways:

① Check LHS = RHS and $x(0) = x_0$ where x_0 is arbitrary
→ will not do this here

② Derive the expression for the solution
→ use integrating factor

Let $x(t) = e^{At} y(t)$ and assume $x(t)$ satisfies $(*)$, then

$$\text{LHS} = \frac{dx}{dt} = \frac{d}{dt}\left(e^{At} y(t)\right) = Ae^{At} y + e^{At} \frac{dy}{dt}$$

$$\text{RHS} = Ax + b(t) = Ae^{At} y + b(t)$$

Hence, $Ae^{At} y + e^{At} \frac{dy}{dt} = Ae^{At} y + b(t)$

so that $e^{At} \frac{dy}{dt} = b(t)$. Multiplying by e^{-At}, we get

$$e^{-At} e^{At} \frac{dy}{dt} = e^{-At} b(t) \quad \text{or} \overset{(*)}{} \quad \frac{dy}{dt} = e^{-At} b(t)$$

Integration gives $y(t) = c + \int_0^t e^{-As} b(s)\, ds$

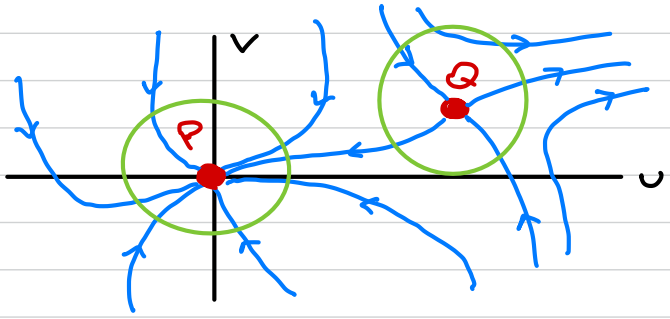
and therefore $x(t) = e^{At} y(t) = e^{At} c + \int_0^t e^{A(t-s)} b(s)\, ds$ □

(*) using the lemma: $e^{-At} e^{At} = e^{(-A+A)t} = e^{0t} = 1$

§4 Nonlinear systems of ODEs and applications

§4.1 Strategy and tools

Goal: design and analyse models of real-world applications



P, Q : equilibria
- P: stable node
- Q: saddle

→
- find equilibria
- determine behavior near equilibria using linear approximation and our linear theory
- piece solutions from different regions together

Need:
- existence and uniqueness (to draw phase portraits)
- tools to compute linear approximations

Focus: nonlinear autonomous systems of ODEs:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} \quad \text{or}$$

$$\frac{dx}{dt} = F(x) \quad \text{with} \quad x = \begin{pmatrix} u \\ v \end{pmatrix}, \; F(x) = F(u,v) = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}$$

Jacobian:

$$\frac{dF}{dx}(x_*) = \frac{dF}{dx}(u_*, v_*) = \begin{pmatrix} \frac{df}{du}(u_*,v_*) & \frac{df}{dv}(u_*,v_*) \\ \frac{dg}{du}(u_*,v_*) & \frac{dg}{dv}(u_*,v_*) \end{pmatrix}$$

2×2 matrix

Example:

$$F(u,v) = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} = \begin{pmatrix} u^2 + uv + e^v \\ \cos v + 2u \end{pmatrix}$$

$$\frac{dF}{dx}(u,v) = \begin{pmatrix} \frac{df}{du} & \frac{df}{dv} \\ \frac{dg}{du} & \frac{dg}{dv} \end{pmatrix} = \begin{pmatrix} 2u+v & u+e^v \\ 2 & -\sin v \end{pmatrix}$$

all entries are continuous in (u,v)

Theorem (existence and uniqueness)

Pick $x_0 \in \mathbb{R}^2$ and assume that there is an $\varepsilon > 0$ so that $F(x)$ and all entries of $\frac{dF}{dx}(x)$ are continuous for all x with $|x - x_0| < \varepsilon$, then the ODE $\dot{x} = F(x)$ with initial condition $x(0) = x_0$ has a unique solution $x(t)$, defined for $t \in (-\delta, \delta)$ for some $\delta > 0$.

Example (continued):

The ODE $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} u^2 + uv + e^v \\ \cos v + 2u \end{pmatrix}$ has a unique solution for each initial condition $(u,v)(0) = (u_0, v_0)$: indeed, the right-hand side and all entries of the Jacobian are continuous for all (u,v) → see above.

Autonomous ODEs:

We focus on autonomous ODEs $\frac{dx}{dt} = F(x)$, where $F(x)$ does not depend on the independent time variable t.

→ the value t_0 of t at which we impose an initial condition does not change the shape of the solution:

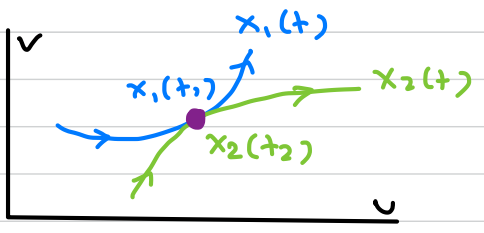
Assume $x_1(t)$ satisfies $\frac{dx_1(t)}{dt} = F(x_1(t))$ with $x_1(0) = x_0$.

Pick $t_0 \in \mathbb{R}$ and define $x_2(t) := x_1(t - t_0)$, then

$$\begin{cases} \frac{dx_2(t)}{dt} = \frac{d}{dt}(x_1(t-t_0)) = \frac{dx_1(t-t_0)}{dt} = F(x_1(t-t_0)) = F(x_2(t)) & \checkmark \\ x_2(t_0) = x_1(t_0 - t_0) = x_1(0) = x_0 & \checkmark \end{cases}$$

Phase portraits:

If $F(x)$ is continuously differentiable for all x, then solutions cannot touch or cross:



$$\begin{cases} \tilde{x}_1(t) = x_1(t + t_1) \\ \tilde{x}_2(t) = x_2(t + t_2) \end{cases}$$

then $\tilde{x}_1(t)$, $\tilde{x}_2(t)$ are solutions with $\tilde{x}_1(0) = \tilde{x}_2(0)$ → contradicts uniqueness!

Equilibria, and their linearizations: Consider $\dot{x} = F(x)$.

- If $F(x_*) = 0$ (so that x_* is a zero or root of $F(x)$), then $x(t) = x_*$ for all t is a time-independent solution or an equilibrium / rest state.

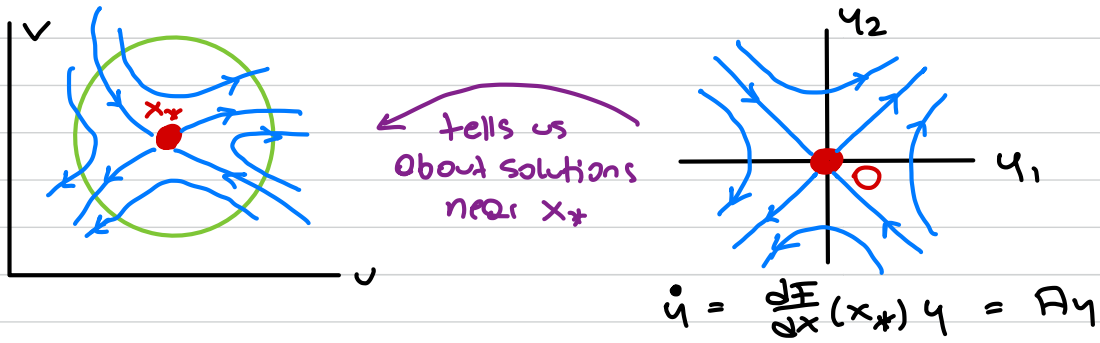
- We want to determine what solutions with initial conditions x_0 near x_* do
  → use Taylor series to approximate $F(x)$ near x_*:

$$F(x_* + y) = \underbrace{F(x_*)}_{=0} + \frac{dF}{dx}(x_*)\,y + \underbrace{\text{terms of least quadratic in } y}$$

smaller than linear terms → omit!

Linearization about x_* is the linear ODE

$$\dot{y} = \frac{dF}{dx}(x_*)\,y = Ay \quad : \text{we know how to solve this!}$$



Theorem (Linearization):
Assume that $F(x_*) = 0$ and that $F(x)$ is continuously differentiable in x for all x near x_*. If all eigenvalues of $\frac{dF}{dx}(x_*)$ have nonzero real part, then the phase portraits of (1) $\dot{x} = F(x)$ near x_* and (2) $\dot{y} = \frac{dF}{dx}(x_*)y$ near $y = 0$ look similar.

§4.2 Ecology : competing species

Question: Will competition lead to coexistence or extinction?

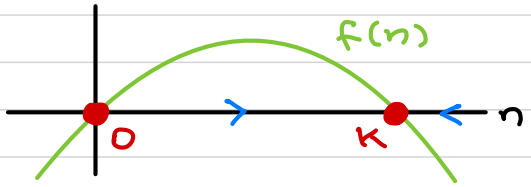
Modelling a single species: logistic model

$n(t)$: population size at time t

model: $\frac{dn}{dt} = an\left(1 - \frac{n}{K}\right) =: f(n)$

where a : growth rate for small populations

K : carrying capacity (accounts for limited resources)



We have
- $n(t) = 0$ for all t or
- $n(t) \to K$ as $t \to \infty$

Modelling two competing species:

$u(t)$: rabbits

$v(t)$: sheep

① Logistic model for each species:

$$\begin{cases} \dot{u} = a u\left(1 - \frac{u}{R}\right) \\ \dot{v} = b v\left(1 - \frac{v}{S}\right) \end{cases}$$

different growth rates and carrying capacities

② Include competition between the two species:

- competition has no effect when $u = 0$ or $v = 0$
  → suggest terms of the form uv
- include such a term in equation for u:

$$\dot{u} = a u\left(1 - \frac{u}{R}\right) - cuv = \frac{a}{R} u\left(R - u - \frac{cR}{a} v\right)$$

minus sign accounts for competition ✓

species v reduces carrying capacity of species u ✓

→ (*) $$\begin{cases} \dot{u} = au\left(1 - \frac{u}{R}\right) - cuv \\ \dot{v} = bv\left(1 - \frac{v}{S}\right) - duv \end{cases}$$

model for two competing species

Case study: focus on specific parameter values

$$\rightarrow \begin{cases} \dot{u} = u(3-2u-v) = 3u-2u^2-uv \\ \dot{v} = v(2-u-v) = 2v-uv-v^2 \end{cases}$$

- equilibria: solve $\begin{cases} u(3-2u-v)=0 \\ v(2-u-v)=0 \end{cases}$

$$(u_*, v_*) = (0,0), (0,2), (3/2,0), (1,1)$$

- classify equilibria: Jacobian $= \begin{pmatrix} 3-4u-v & -u \\ -v & 2-u-2v \end{pmatrix}$ evaluate at each equilibrium

$(0,0)$: $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ unstable node

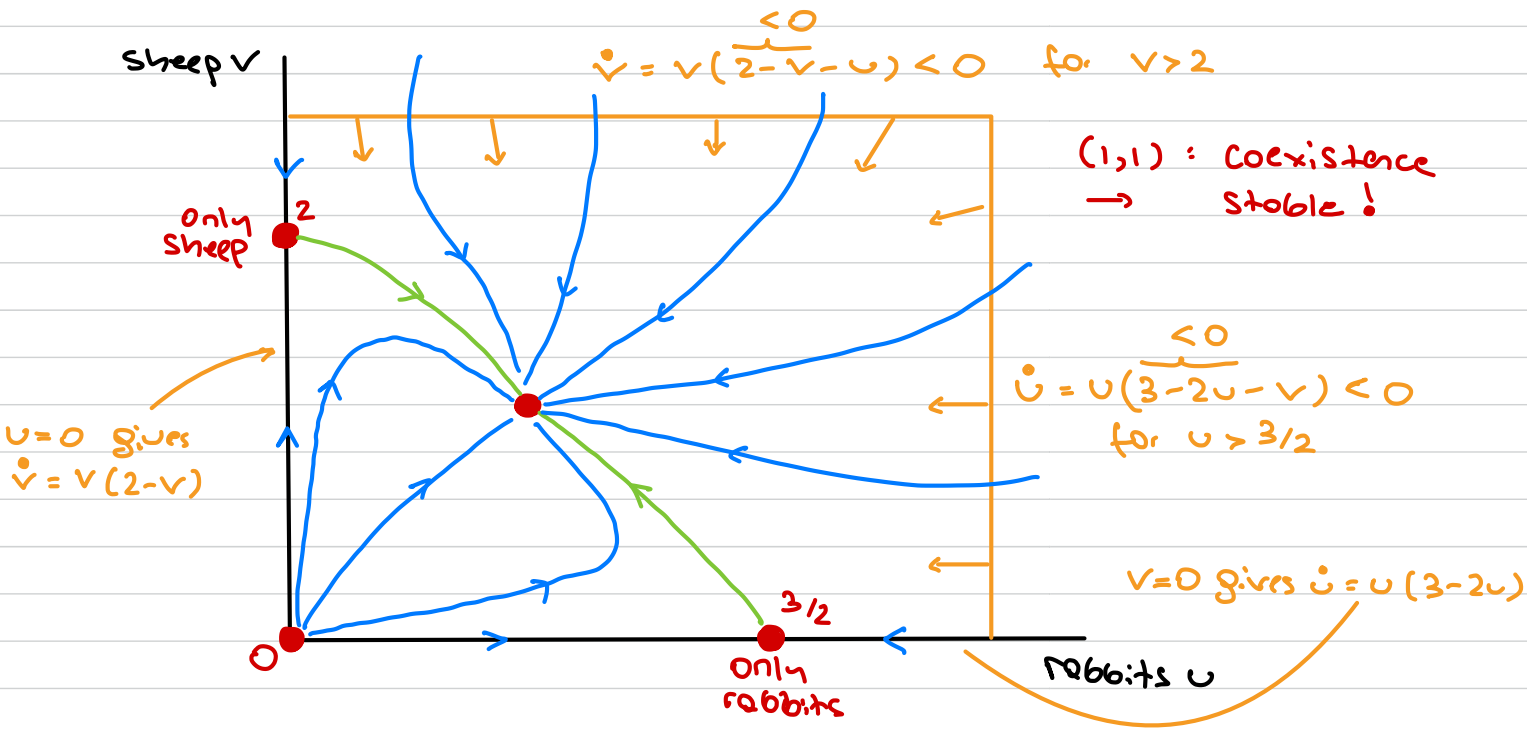
$(0,2)$: $\begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}$ saddle

$(3/2,0)$: $\begin{pmatrix} -3 & -3/2 \\ 0 & 1/2 \end{pmatrix}$ saddle

$(1,1)$: $\begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$ has eigenvalues $\lambda_{1,2} = \frac{-3 \pm \sqrt{5}}{2} < 0$ → stable node

- phase portrait:

(*) see next page



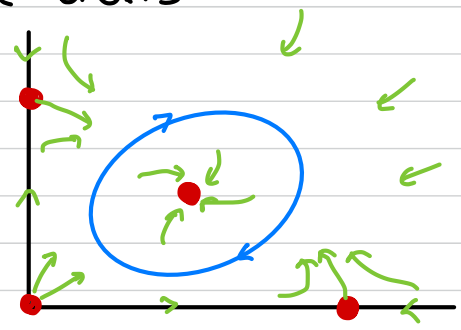
- conclusion: presence of both species leads to coexistence; neither species becomes extinct

(*)

- existence and uniqueness:
  to meaningfully draw the phase portrait, we need to know that solutions exist and are unique once we specify initial conditions

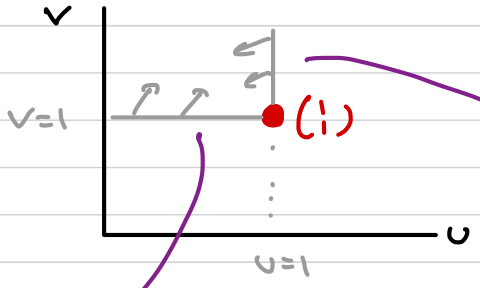
  → our model satisfies the hypotheses of the existence and uniqueness theorem, which therefore guarantees existence and uniqueness:
  indeed, the RHS and all entries of the Jacobian computed above are continuous in (u,v) for all u,v.

- periodic orbits:



to conclude that all solutions in the first quadrant that do not lie on the u- or v-axis converge to the coexistence state, we need to exclude the existence of periodic solutions (shown in blue)

→ this can be done by showing the following:



this can be done as above:

$\dot{u} = u(3-2u-v) \overset{u=1}{=} 1\cdot(3-2-v) = 1-v < 0$ for $v>1$ ✓

$\dot{v} = v(2-u-v) \overset{v=1}{=} 1(2-u-1) = 2-u > 0$ for $0<u<1$ ✓

§4.3 Epidemiology: modelling infectious diseases

Focus on infectious diseases such as Covid-19:

- viral infection
- transmitted by contact between individuals
- confer short-term immunity once antibodies have been produced

Questions we want to address:

- when do outbreaks occur?
- can we predict incidence curves?
- managing outbreaks: „flattening the curve"
- preventing outbreaks: efficacy of vaccinations

Approach:

- population-based model for { S susceptible; I infected; R recovered

Compartment model:

We will make the following assumptions:

- infected = infectious (no latency)
- will not account for population changes due to births/deaths
- recovered are immune



F_2 Assume a constant fraction γ of the infected population recovers per unit time $\longrightarrow$ $F_2 = \gamma I$

note: We can think of $1/\gamma$ as the mean residence time of the disease: if the fraction γ of an individual recovers per unit time, then the individual recovered fully after $1/\gamma$ time units (this argument does not make sense practically but is valid mathematically)

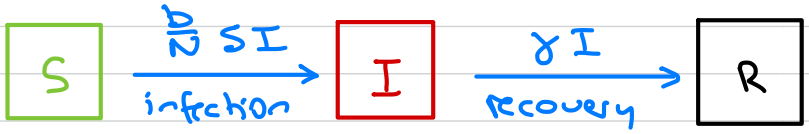
F_1

- assume that each infected individual posses the pathogen to b other individuals per unit time
- if we ignore that two or more infected individuals may poss the pathogen to the some individuals, then pathogens are passed to bI unique individuals per unit time
- transmission affects only susceptible individuals, that is the fraction S/N of the total population of size N
- if individuals interact regardless of the compartment they belong to, we obtain

$$F_2 = \underbrace{bI \cdot S/N} = \frac{b}{N} SI$$

= number of susceptibles who are infected by infected population per unit time

Updated compartment model:



ODE model:

$$\begin{cases} \frac{dS}{dt} = -\frac{b}{N} SI & \text{susceptible} \\ \frac{dI}{dt} = \frac{b}{N} SI - \gamma I & \text{infected} \\ \frac{dR}{dt} = \gamma I & \text{recoverd} \end{cases}$$

① Total population size:

We have $N = S+I+R$ and therefore

$$\frac{dN}{dt} = \frac{d}{dt}(S+I+R) = \dot{S} + \dot{I} + \dot{R} = -\frac{b}{N} SI + \frac{b}{N} SI - \gamma I + \gamma I = 0 \quad \checkmark$$

and we conclude that the total population is constant.
$\longrightarrow$ we can focus on (S,I) since $R = N - S - I$.

② Reduced model:

$$\begin{cases} \dot{S} = -\frac{b}{N} SI \\ \dot{I} = \frac{b}{N} SI - \gamma I \end{cases}$$

three parameters: b, γ, N

③ Nondimensionalization:

- normalize time via $\tau := \gamma t$
  so that $\tau = 1$ corresponds to $t = 1/\gamma$ = mean residence time of disease
- convert S, I to fractions u, v of total population N:

$$\begin{cases} u(\tau) = u(\gamma t) := \frac{S(t)}{N} & \text{or} \quad S(t) = N u(\gamma t) \\ v(\tau) = v(\gamma t) := \frac{I(t)}{N} & \text{or} \quad I(t) = N v(\gamma t) \end{cases}$$

- derive ODEs for $(u, v)(\tau)$:

$$\text{LHS} = \frac{dS}{dt} = \frac{d}{dt}(N u(\gamma t)) = N \frac{du(\tau)}{d\tau} \frac{d(\gamma t)}{dt} = \gamma N \frac{du}{d\tau}$$

(independent of t; chain rule)

$$\text{RHS} = -\frac{b}{N} SI = -bN \frac{S}{N} \frac{I}{N} = -bN\, uv$$

$$\Rightarrow \frac{du}{d\tau} = -\frac{b}{\gamma} uv$$ fraction of population that is susceptible

$$\text{LHS} = \frac{dI}{dt} = \frac{d}{dt}(N v(\gamma t)) = N \frac{dv(\tau)}{d\tau} \frac{d(\gamma t)}{dt} = \gamma N \frac{dv}{d\tau}$$

$$\text{RHS} = \frac{b}{N} SI - \gamma I = bN \frac{S}{N} \frac{I}{N} - \gamma N \frac{I}{N} = bN uv - \gamma N v$$

$$\Rightarrow \frac{dv}{d\tau} = \frac{b}{\gamma} uv - v$$ fraction of population that is infected

- result:

$$(*) \quad \begin{cases} \frac{du}{d\tau} = -\frac{b}{\gamma} uv \\ \frac{dv}{d\tau} = \frac{b}{\gamma} uv - v \end{cases}$$

new rescaled model

$\longrightarrow$ define new parameter $R_0 := \frac{b}{\gamma}$ basic reproduction number

④ Existence and uniqueness:

The right-hand side and all its derivatives in (u, v) are continuous for all (u, v), so existence and uniqueness of solutions is assured by the existence and uniqueness theorem.

⑤ SIR model :

$$(*)\quad \begin{cases} \frac{du}{d\tau} = -R_0 uv \\ \frac{dv}{d\tau} = R_0 uv - v \end{cases}$$

where $0 \le u, v \le 1$
$R_0 = b/\gamma$ parameter

⑥ Phase portrait of (*) :

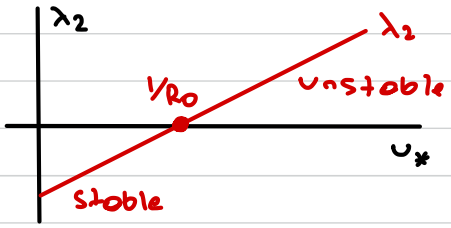
- equilibria : $\dot{u}=0$ and $\dot{v}=0$ give $\begin{cases} -R_0 uv = 0 \\ R_0 uv - v = 0 \end{cases}$

  hence $(u,v) = (u_*, 0)$: rest states where u_* is arbitrary

- Jacobian : $\begin{pmatrix} -R_0 v & -R_0 u \\ R_0 v & R_0 u - 1 \end{pmatrix}\Big|_{(u_*,0)} = \begin{pmatrix} 0 & -R_0 u_* \\ 0 & R_0 u_* - 1 \end{pmatrix}$

  $\longrightarrow$ $\lambda_1 = 0$ neutral

  $\lambda_2 = R_0 u_* - 1$ $\begin{cases} <0 & \text{for } u_* < \frac{1}{R_0} \quad \text{stable} \\ >0 & \text{for } u_* > \frac{1}{R_0} \quad \text{unstable} \end{cases}$



- initial conditions : Recall $S+I+R=N$ $\Rightarrow$ $\frac{S}{N} + \frac{I}{N} + \frac{R}{N} = 1$

  Hence $\boxed{u + v + \frac{\#\text{recovered/vaccinated}}{N} = 1}$

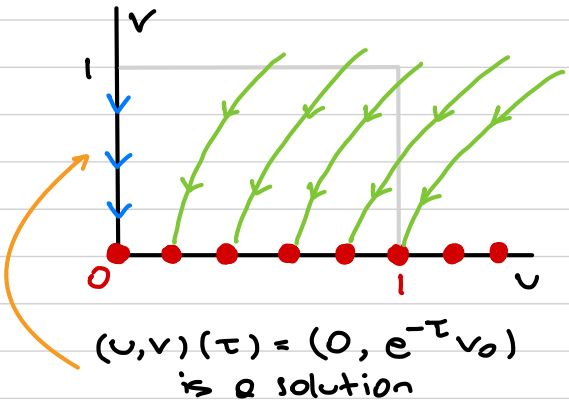
  - new disease : $(u,v)(0) \approx (1,0)$ few infected individuals in otherwise susceptible populations for new disease (no vaccines ; no recovered)

  - established disease : $(u,v)(0) = (u_*, v_*)$ with $u_* + v_* < 1$ $\quad 1 - u_* - v_*$ recovered or vaccinated fraction of population

    $\longrightarrow$ herd immunity ?

$R_0 < 1$

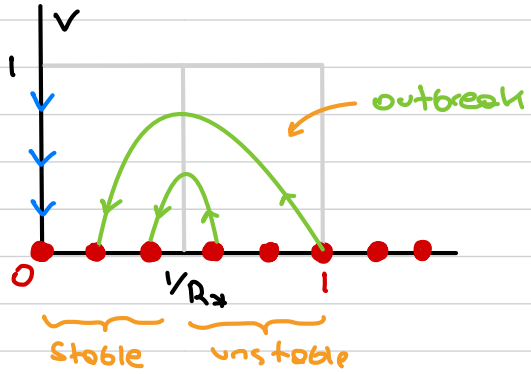
$(v_*, 0)$ with $0 \le v_* \le 1$ have $\lambda_1 = 0$ and $\lambda_2 < 0$



→ $R_0 < 1$: no outbreak

$R_0 > 1$

$(v_*, 0)$ with $0 \le v_* \le 1$ have $\lambda_1 = 0$ and $\lambda_2 \begin{cases} < 0 & v_* < 1/R_0 \\ > 0 & 1/R_0 < v_* \le 1 \end{cases}$



→ $R_0 > 1$: leads to outbreak if entire population is susceptible

(7) Conclusions:

- **Outbreaks** occur when $R_0 > 1$ for new diseases without recovered / immune / vaccinated population

- **vaccination / herd immunity:**

  Claim: $r_* = \frac{R}{N}$ = fraction of vaccinated / recovered $> 1 - 1/R_0$ implies $v_* < 1/R_0$, and outbreak disappears

  Proof: $r_* > 1 - 1/R_0$ gives $-r_* < 1 - 1/R_0$ and therefore

  $v_* = 1 - v_* - r_* \le 1 - r_* < 1 - (1 - 1/R_0) = 1/R_0$ ✓

  → NYT data from spring 2020 indicate $R_0 \approx 3.8$ for COVID-19 so that $\ge 75\%$ $(1 - 1/R_0 \approx 0.74)$ of the population needs to be vaccinated / immune from COVID-19

- **interpretation of R_0:**

  assume single infected individual in population of susceptibles:

  $R_0 = b \cdot 1/\gamma = \frac{\#\text{ infected individuals}}{\text{unit time}} \times \text{duration of infection}$

  $= \#$ individuals infected whilst individual is infectious

  → $R_0 > 1$ implies that each infected individuals infects at least one other person over the course of being sick

  → infected population increases!

⑧ Justification for phase portraits

- $\frac{dv}{dt} = -R_0 uv < 0$ unless $uv = 0$ $\Rightarrow$ $u(t)$ decreases strictly
- $\frac{du}{dt} = (R_0 u - 1)v$ $\begin{cases} <0 & u < 1/R_0 \\ =0 & u = 1/R_0 \\ >0 & u > 1/R_0 \end{cases}$ (for all $u \leq 1$ when $R_0 < 1$)
- $\frac{d}{dt}(u+v) = -v < 0$ $\Rightarrow$ $u(t) + v(t) \leq 1$ for all $t \geq 0$

§4.4 Climate : energy-balance models

not done in class

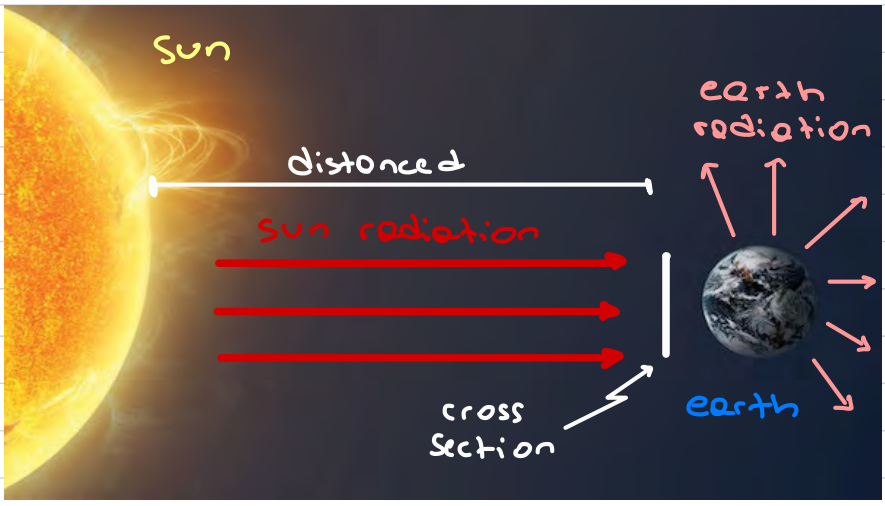
Question: What determines the surface temperature of planets ?

Goals:
- consider energy-balance model for planets
- investigate transitions between states via tipping points

Radiation: Sun acts as a blackbody : it emits electromagnetic waves across a broad spectrum of wavenumbers with spectrum and energy output depending on its effective surface temperature T_s

Earth reflects a fraction α of the radiation reaching it from the sun and absorbs the remaining fraction $1-\alpha$; The albedo α depends on surface conditions such as area covered by oceans, ice, forests etc.;

Earth also acts as a blackbody and emits radiation whose energy depends on the surface temperature T_e and the amount and properties of greenhouse gases in the atmosphere



r_s radius of sun
r_e radius of earth
d sun-earth distance

Physics:
- energy flux is defined by energy per unit area and unit time
- quantum mechanics predicts the energy flux emitted from blackbodies with a given temperature

not done in class

Radiation emission and absorption:

- solar energy flux reaching earth is given by the solar constant $L := \sigma T_s^4 r_s^2 / d^2$ where σ is the Stefan-Boltzmann constant

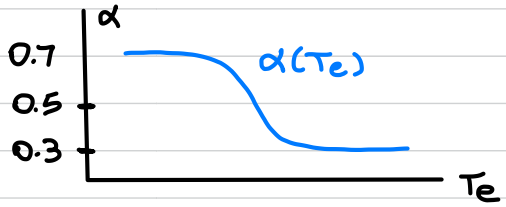
- earth therefore absorbs the total energy $\pi r_e^2 (1-\alpha) L$ per unit time, where α is the albedo.

- earth emits the total energy $4\pi r_e^2 (1-\varepsilon) \sigma T_e^4$ per unit time: $\varepsilon = 0$ corresponds to the absence of greenhouse gases, while $\varepsilon > 0$ reflects increased reflection of emitted blackbody radiation due to greenhouse gases in the atmosphere

Albedo-climate feedback:

- albedo α depends on surface temperature T_e of earth:
  temperature decreases $\Rightarrow$ area covered by ice increases
  $\Rightarrow$ fraction α of radiation reflected increases

- model:
$$\alpha(T_e) = \frac{1}{2} - \frac{1}{5} \tanh\left(\frac{T_e - 260}{10}\right)$$



Energy-balance model:

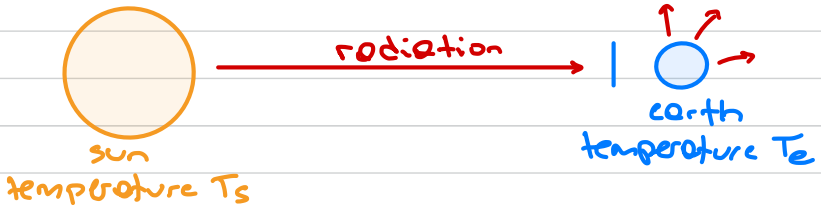
- $E(T_e)$: surface energy storage per unit area at temperature T_e
- compartment model:

$$\frac{d}{dt}(E(T_e)) = \text{energy per unit area absorbed by earth per unit time} - \text{energy per unit area emitted by earth per unit time}$$
$$= \frac{1}{4}(1-\alpha(T_e)) L - (1-\varepsilon)\sigma T_e^4$$

where we used $4\pi r_e^2$ = area of earth surface to calculate the fluxes on the right-hand side

- $$\frac{d}{dt}(E(T_e)) = \underbrace{\frac{dE}{dT_e}(T_e)}_{=: \gamma(T_e)\text{: heat capacity}} \frac{dT_e}{dt}$$

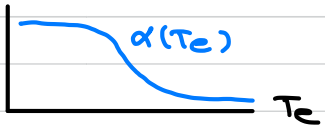
Abbreviated „climate energy-balance model" notes



r_s radius of sun
r_e radius of earth
d sun-earth distance

- energy flux = energy per unit area and per unit time
- blackbody radiation: energy flux at surface = σT^4
- energy flux emitted by sun reaching earth: $\sigma T_s^4 \, r_s^2/d^2 =: L$
- total energy absorbed by earth: $\pi r_e^2 (1-\alpha) L$ α: albedo
- total energy emitted by earth: $4\pi r_e^2 (1-\varepsilon) \sigma T_e^4$
  where $\varepsilon \approx$ greenhouse gas effect in atmosphere
- energy stored on surface per unit area: $E(T_e)$
- albedo: $\alpha = \alpha(T_e)$ where

$$\alpha(T_e) = \frac{1}{2} - \frac{1}{5} \tanh\left(\frac{T_e - 260}{10}\right)$$



$$\longrightarrow \quad \underbrace{\frac{d}{dt}(E(T_e))}_{\frac{dE}{dT}(T_e)\cdot\frac{dT_e}{dt}} = \underbrace{E_{in} - E_{out}}_{\text{fluxes}} = \frac{1}{4}(1-\alpha) L - (1-\varepsilon)\sigma T_e^4$$

$$\longrightarrow \quad \frac{dT_e}{dt} = \frac{1}{C}\left[\frac{L}{4}(1-\alpha(T_e)) - (1-\varepsilon)\sigma T_e^4\right]$$

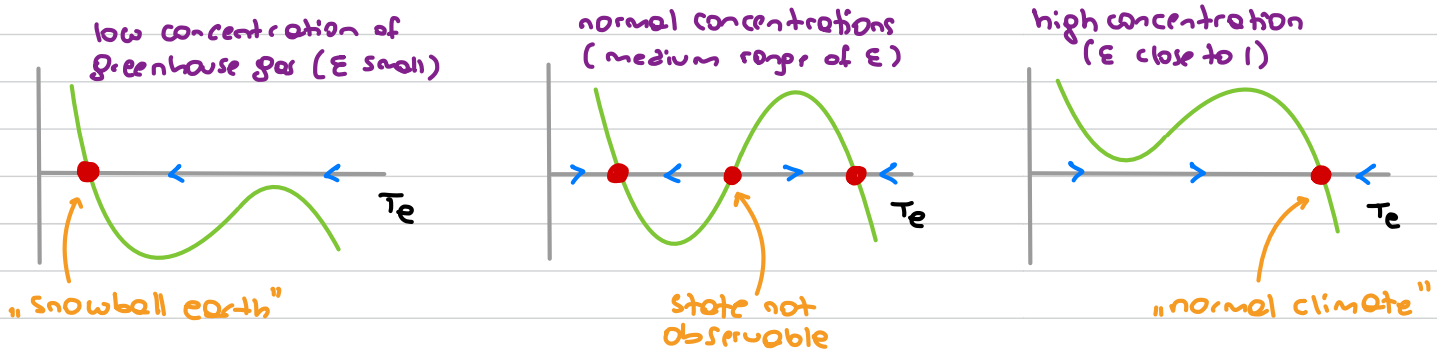
where we assume $C = \frac{dE}{dT}(T_e)$: heat capacity is constant

Model: $\frac{dT_e}{dt} = \frac{1}{C}\left[(1-\alpha(T_e))\frac{L}{4} - (1-\varepsilon)\sigma T_e^4\right]$

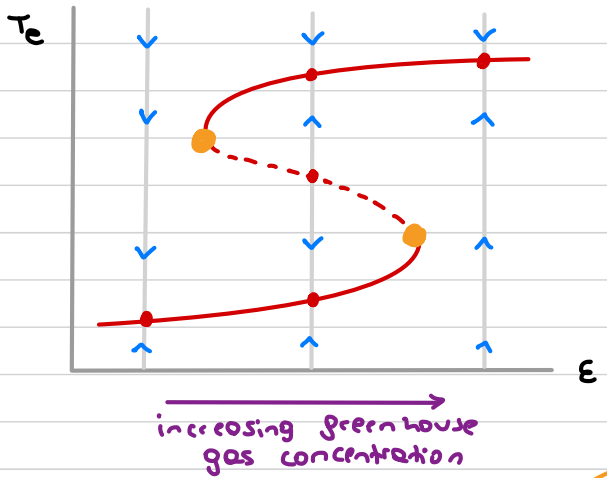
$\alpha(T_e) = \frac{1}{2} - \frac{1}{5}\tanh\left(\frac{T_e - 260}{10}\right)$

where we assumed (for simplicity) that $C := \rho(T_e)$ does not depend on T_e

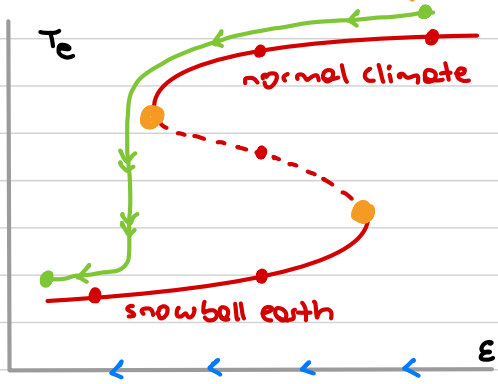
Phase portraits: I will sketch phase portrait for different values of ε that I obtained using a Jupyter notebook; we can obtain these results analytically using qualitative analysis!



Tipping points: use (ε, T_e) plane to represent the phase portraits



- tipping points: two equilibria collide and disappear as ε is varied



T_e and ε depend both on time

assume $\varepsilon = \varepsilon(t)$ decreases slowly over time → T_e follows normal climate state until this state disappears and then rapidly drops to snowball earth state!

§5 Second-order equations

Focus on linear second-order equations in the form

$$(*) \quad \frac{d^2u}{dt^2} + p\frac{du}{dt} + qu = 0 \qquad \text{for given constants } p, q \in \mathbb{R}.$$

$\longrightarrow$ serve as models for
- mechanical systems (mass-spring, linear oscillators, ...)
- electric circuits (capacitors, resistors, inductors)

• Equivalence to first-order systems

Consider $(1) \quad \frac{d^2u}{dt^2} = F\left(t, u, \frac{du}{dt}\right)$
where we assume that $F(t,u,v)$ is continuously differentiable in (u,v).

Let $v := \frac{du}{dt}$, then (1) can be written as the system

$$(2) \quad \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ F(t,u,v) \end{pmatrix}$$

Lemma 1:
1. $u(t)$ satisfies (1) if and only if $(u,v)(t) = \left(u(t), \frac{du}{dt}(t)\right)$ satisfies (2).
2. Assume $F(t,u,v)$ is continuously differentiable in (u,v) and continuous in t. For each given (t_0, u_0, v_0), the ODE (1) with initial condition $u(t_0) = u_0$ and $\frac{du}{dt}(t_0) = v_0$ has a unique solution.

• Solutions to linear second-order equations

Next, consider $(3) \quad \frac{d^2u}{dt^2} + p\frac{du}{dt} + qu = 0$ for constants p, q,
then the associated first-order system is given by

$$(4) \quad \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} =: A\begin{pmatrix} u \\ v \end{pmatrix}$$

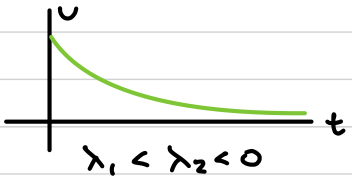
Lemma 2: The following statements are equivalent:
1. λ is an eigenvalue of $A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}$, with eigenvector $x = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$
2. $\lambda^2 + p\lambda + q = 0$
3. $u(t) = e^{\lambda t}$ satisfies (3)

We can now use Lemmas 1 and 2 together with our results on first-order systems to obtain the following characterization of solutions to (3):

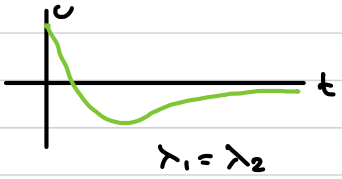
Lemma 3: Consider (*) $\frac{d^2u}{dt^2} + p\frac{du}{dt} + qu = 0$ for constants $p, q \in \mathbb{R}$. Denote by $\lambda_{1,2}$ the roots of $\lambda^2 + p\lambda + q = 0$. Then every solution $u(t)$ of (*) is of the following form for appropriate constants c_1, c_2:

$$u(t) = \begin{cases} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} & \text{for } \lambda_1 \neq \lambda_2 \in \mathbb{R} \\ c_1 e^{\lambda t} + c_2 t e^{\lambda t} & \text{for } \lambda_1 = \lambda_2 = \lambda \\ c_1 e^{rt}\cos \omega t + c_2 e^{rt} \sin \omega t & \text{for } \lambda_{1,2} = r \pm i\omega \notin \mathbb{R} \end{cases}$$

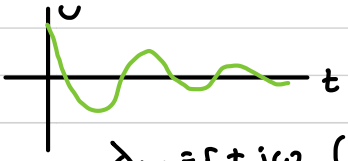
Furthermore, given constants u_0, v_0, there are unique constants c_1, c_2 so that $u(t)$ satisfies (*) with $u(0) = u_0$, $\frac{du}{dt}(0) = v_0$



$\lambda_1 < \lambda_2 < 0$



$\lambda_1 = \lambda_2$



$\lambda_{1,2} = r \pm i\omega$ $(\omega \neq 0)$ with $r < 0$

Example 1: Find all solutions to $\frac{d^2u}{dx^2} = au$ where $a \in \mathbb{R}$ is given.

→ The roots of $\lambda^2 = a$ are given by $\lambda_{1,2} = \begin{cases} \pm\sqrt{a} & a > 0 \\ 0 & a = 0 \\ \pm i\sqrt{|a|} & a < 0 \end{cases}$

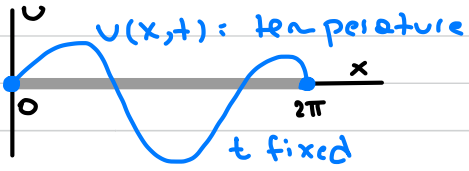
Hence the solutions are given by

$$u(x) = \begin{cases} c_1 e^{-\sqrt{a}x} + c_2 e^{\sqrt{a}x} & a > 0 \\ c_1 + c_2 x & a = 0 \\ c_1 \cos(\sqrt{|a|}\,x) + c_2 \sin(\sqrt{|a|}\,x) & a < 0 \end{cases}$$ ✓

- **Boundary-value problems**

We consider the temperature profile along a beam of length 2π, where we hold the temperature at the end points fixed at zero. The heat equation with Dirichlet boundary conditions given by

$$(S) \quad \begin{cases} \frac{du}{dt} = \frac{d^2u}{dx^2} & t \geq 0, \quad 0 < x < 2\pi \\ u(0,t) = 0 & t \geq 0 \\ u(2\pi, t) = 0 & t \geq 0 \end{cases}$$



models the temperature profile $u(x,t)$ along the beam (see APMA 0360).

We will use separation of variables and seek solutions of the form $u(x,t) = e^{\lambda t} v(x)$ where $\lambda \in \mathbb{R}$ and $v(x)$ is not identically equal to zero:

① $\begin{cases} \text{LHS} = \frac{dv}{dt} = \frac{d}{dt}\left(e^{\lambda t} v(x)\right) = \lambda e^{\lambda t} v(x) \\ \text{RHS} = \frac{d^2v}{dx^2} = \frac{d^2}{dx^2}\left(e^{\lambda t} v(x)\right) = e^{\lambda t} \frac{d^2v}{dx^2}(x) \end{cases}$

LHS = RHS gives $\lambda e^{\lambda t} v(x) = e^{\lambda t} \frac{d^2v}{dx^2}(x)$
and $v(x)$ therefore needs to satisfy the ODE $\frac{d^2v}{dx^2} = \lambda v$
where $\lambda \in \mathbb{R}$ is undetermined.

② It remains to address the boundary conditions:

$\begin{cases} u(0,t) = \left(e^{\lambda t} v(x)\right)|_{x=0} = e^{\lambda t} v(0) = 0 \\ u(2\pi,t) = \left(e^{\lambda t} v(x)\right)|_{x=2\pi} = e^{\lambda t} v(2\pi) = 0 \end{cases}$ for all $t \geq 0$

and we see that this holds if and only if $v(0) = 0$ and $v(2\pi) = 0$

③ Hence, finding solutions $u(x,t) = e^{\lambda t} v(x)$ of (5) reduces to finding solutions $v(x)$ of

$\begin{cases} (6)\ \frac{d^2v}{dx^2} = \lambda v \qquad 0 < x < 2\pi \\ (7)\ v(0) = v(2\pi) = 0 \end{cases}$

Example 1 shows that all solutions to (6) are given by

$$v(x) = \begin{cases} c_1 e^{-\sqrt{\lambda}x} + c_2 e^{\sqrt{\lambda}x} & \lambda > 0 \\ c_1 + c_2 x & \lambda = 0 \\ c_1 \cos(\sqrt{|\lambda|}x) + c_2 \sin(\sqrt{|\lambda|}x) & \lambda < 0 \end{cases}$$

We need to check the boundary conditions (7) for each case:

$\lambda = 0$: $v(x) = c_1 + c_2 x$ and therefore

$\begin{cases} 0 = v(0) = c_1 & \text{hence } c_1 = 0 \\ 0 = v(2\pi) = c_1 + 2\pi c_2 = 2\pi c_2 & \text{hence } c_2 = 0 \end{cases}$

$\lambda > 0$: $v(x) = c_1 e^{-\sqrt{\lambda}x} + c_2 e^{\sqrt{\lambda}x}$ and therefore

$\begin{cases} 0 = v(0) = c_1 + c_2 \\ 0 = v(2\pi) = c_1 e^{-2\pi\sqrt{\lambda}} + c_2 e^{2\pi\sqrt{\lambda}} \end{cases}$ or $\begin{pmatrix} 1 & 1 \\ e^{-2\pi\sqrt{\lambda}} & e^{2\pi\sqrt{\lambda}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$

which has a nonzero solution (c_1, c_2) if and only if

$$0 = \begin{vmatrix} 1 & 1 \\ e^{-2\pi\sqrt{\lambda}} & e^{2\pi\sqrt{\lambda}} \end{vmatrix} = e^{2\pi\sqrt{\lambda}} - e^{-2\pi\sqrt{\lambda}}$$

which holds only for $\lambda = 0$ which is excluded since $\lambda > 0$.

$\lambda < 0$: $v(x) = c_1 \cos(\sqrt{|\lambda|}x) + c_2 \sin(\sqrt{|\lambda|}x)$ hence

$\begin{cases} 0 = v(0) = c_1 & \text{hence } c_1 = 0 \\ 0 = v(2\pi) = c_1 \cos(2\pi\sqrt{|\lambda|}) + c_2 \sin(2\pi\sqrt{|\lambda|}) = c_2 \sin(2\pi\sqrt{|\lambda|}) \end{cases}$ ($c_1 = 0$)

Hence, we have $c_2 = 0$ or need $\sin(2\pi\sqrt{|\lambda|}) = 0$
which requires $2\pi\sqrt{|\lambda|} = n\pi$ or $|\lambda| = n^2/4$ for integers n.
Since $\lambda < 0$, we obtain $\lambda = -n^2/4$ for integers $n \geq 1$.

③ Conclusion: The only solutions of equation (1) of the form $u(x,t) = e^{\lambda t} v(x)$ with $\lambda \in \mathbb{R}$ and $v(x) \not\equiv 0$ are $u_n(x,t) = e^{-n^2 t/4} \sin(2\pi n x)$ for integers $n \geq 1$.

• Inhomogeneous equations (and resonance)

Consider the inhomogeneous linear ODE

$$(8) \quad \frac{d^2u}{dt^2} + p\frac{du}{dt} + qu = g(t) \qquad p, q \text{ constants}$$

Lemma 4: Assume that $g(t)$ is continuous for all t.

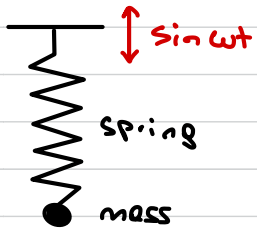
① If • $u(t) = c_1 u_1(t) + c_2 u_2(t)$ is the general solution of the homogeneous ODE $\frac{d^2u}{dt^2} + p\frac{du}{dt} + qu = 0$ from Lemma 3, and
• $u_p(t)$ is a solution of the inhomogeneous ODE (8)
then $u(t) = c_1 u_1(t) + c_2 u_2(t) + u_p(t)$ is the general solution of (8).

② If $g(t) = g_1(t) + \dots + g_N(t)$ where each term is continuous in t and $u_p^{(j)}(t)$ are solutions of $\frac{d^2u}{dt^2} + p\frac{du}{dt} + qu = g_j(t)$ for $j = 1, \dots, N$, then $u_p(t) = u_p^{(1)}(t) + \dots + u_p^{(N)}(t)$ is a solution of (8).

Note: $u_p(t)$ is called a particular solution of (8): we can use any solution of (8) for it, and Lemma 4 then allows use to assemble the general solution of (8) based on the particular solution of (8) and the general solution of the homogeneous equation associated with (8) (which has $g(t) = 0$).

Example 2: Solve $\frac{d^2u}{dt^2} + u = \sin(\omega t)$.

→ models a linear pendulum or mass-spring system that is subjected to periodic forcing with frequency $\omega > 0$.



① Solve homogeneous equation (*) $\frac{d^2u}{dt^2} + u = 0$.
Substituting $u(t) = e^{\lambda t}$ gives $\lambda^2 + 1 = 0$ hence $\lambda = \pm i$.
Hence, $u(t) = c_1 \cos t + c_2 \sin t$ is the general solution of the homogeneous equation (*).

② Next, we need to find a particular solution of

$$(**) \quad \frac{d^2u}{dt^2} + u = \sin(\omega t)$$

Note: exponential functions (including cos and sin) reproduce themselves under differentiation
→ seek solution in the same form as forcing term

The approach is called „undetermined coefficients":
We try $v(t) = A\cos\omega t + B\sin\omega t$, where A, B are constants that we need to determine. We need LHS = RHS:

- LHS = $\frac{d^2v}{dt^2} + v = -A\omega^2\cos\omega t - B\omega^2\sin\omega t + A\cos\omega t + B\sin\omega t$
- RHS = $\sin\omega t$
- Need $A(1-\omega^2)\cos\omega t + B(1-\omega^2)\sin\omega t = \sin\omega t$ for all t

We have LHS = RHS for all t if $A = 0$ and $B = \frac{1}{1-\omega^2}$ which is possible when $\omega \neq \pm 1$.

For $\omega \neq \pm 1$, $v_p(t) = \frac{1}{1-\omega^2}\sin\omega t$ is a particular solution

③ For $\omega \neq \pm 1$, the general solution of our model is given by
$v(t) = c_1\cos t + c_2\sin t + \frac{1}{1-\omega^2}\sin\omega t$

④ Resonance $\omega = 1$: forcing has same frequency as the unforced mechanical system!

modify „undetermined coefficients" guess:
Let $v(t) = At\cos t + Bt\sin t$, then

$$\frac{dv}{dt} = A\cos t + B\sin t - At\sin t + Bt\cos t$$
$$\frac{d^2v}{dt^2} = -A\sin t + B\cos t - A\sin t + B\cos t - At\cos t - Bt\sin t$$
$$= -2A\sin t + 2B\cos t - At\cos t - Bt\sin t$$

- LHS = $\frac{d^2v}{dt^2} + v = -2A\sin t + 2B\cos t - At\cos t - At\sin t + At\cos t + Bt\sin t = -2A\sin t + 2B\cos t$
- RHS = $\sin t$
- need $-2A\sin t + 2B\cos t = \sin t$ for all t

hence, $A = -1/2$ and $B = 0$

Hence, $v_p(t) = -\frac{t}{2}\cos t$ satisfies our model for $\omega = 1$

$\Rightarrow$ $v(t) = c_1\cos t + c_2\sin t - \frac{t}{2}\cos t$ is the general solution when $\omega = 1$

($-\frac{t}{2}\cos t$: unbounded growth in t!) ✓

Finding particular solutions :

① Undetermined coefficients :

$$g(t) = \begin{cases} e^{at} & \rightarrow \quad v(t) = Ae^{at} \\ & \quad (\text{or } v(t) = Ate^{at} \text{ if } \lambda_1 = a \neq \lambda_2) \\ \cos bt \text{ or } \sin bt & \rightarrow \quad v(t) = A\cos bt + B\sin bt \\ & \quad (\text{or } v(t) = At\cos bt + Bt\sin bt \text{ if } \lambda = \pm ib) \end{cases}$$

see [Logan : §2.3] for other right-hand sides

② Variation of constants formula :

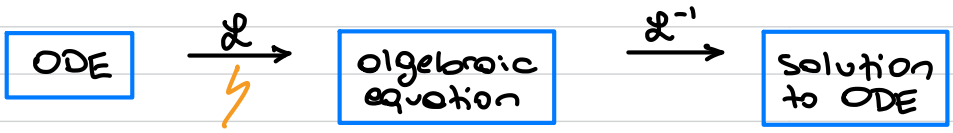
Apply our first-order formula from §3.3 [Logan : §2.4.2] :

$$\begin{cases} v_p(t) = -v_1(t)\int_0^t \frac{v_2(s)\,g(s)}{W(s)}\,ds + v_2(t)\int_0^t \frac{v_1(s)\,g(s)}{W(s)}\,ds \\ W(t) := v_1(t)\frac{dv_2}{dt}(t) - \frac{dv_1}{dt}(t)\,v_2(t) \end{cases}$$

where $v(t) = c_1 v_1(t) + c_2 v_2(t)$ denotes the general solution of the associated homogeneous problem.

§6 Laplace transform

- **Outline**
  - We will learn about integral transforms, which provide a very different approach to solving linear ODEs.
  - Integral transforms, such as Laplace transform and Fourier transform, are especially useful for solving PDEs!
  - They provide an alternative way to solving inhomogeneous equations, especially for step functions and impulse forcing
  - Idea: transform ODE into algebraic equation



$\mathcal{L}$ = Laplace transform

- **Definition and properties**

Definition: If $v(t)$ is a continuous function defined for $t \geq 0$, its Laplace transform $V(s) = [\mathcal{L}v](s)$ is the function

$$V(s) = \int_0^\infty v(t)\, e^{-st}\, dt \quad \text{provided} \quad \lim_{T\to\infty} \int_0^T v(t)\, e^{-st}\, dt \quad \text{exists.}$$

Example: Let $v(t) = e^{at}$, then

$$V(s) = \mathcal{L}(e^{at}) = \int_0^\infty v(t) e^{-st}\, dt = \int_0^\infty e^{at} e^{-st}\, dt$$

$$= \int_0^\infty e^{(a-s)t}\, dt = \frac{1}{a-s}\, e^{(a-s)t}\Big|_{t=0}^{t=\infty} = \frac{1}{s-a}$$

↳ need $s>a$ at $t=\infty$

defined for $s > a$.

Hence, $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ or $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

Inverse Laplace transform: We write $v = \mathcal{L}^{-1}(V)$ if $V = \mathcal{L}(v)$

→
- no explicit formula to compute $\mathcal{L}^{-1}(V)$!
- will normally use lookup tables:

v	V
e^{at}	$\frac{1}{s-a}$

Linearity:

- $\mathcal{L}(c_1 v_1 + c_2 v_2) = c_1 \mathcal{L}(v_1) + c_2 \mathcal{L}(v_2)$
- $\mathcal{L}^{-1}(c_1 V_1 + c_2 V_2) = c_1 \mathcal{L}^{-1}(V_1) + c_2 \mathcal{L}^{-1}(V_2)$

} for $c_1, c_2 \in \mathbb{R}$

- **Differential equations**

Example: Solve $\frac{dv}{dt} = -v$ with $v(0) = 2$ using Laplace transform

$\longrightarrow \quad \mathcal{L}\left(\frac{dv}{dt}\right) = -\mathcal{L}(v)$

$\int_0^\infty \frac{dv}{dt}(t)\, e^{-st}\, dt = -\mathcal{L}(v)$: integrate by parts

$= \lim_{T\to\infty} \int_0^T \frac{dv}{dt}(t)\, e^{-st}\, dt$

$= \lim_{T\to\infty} \left[(v(t)\, e^{-st})\Big|_{t=0}^{t=T} - \int_0^T v(t)\, \frac{d}{dt}(e^{-st})\, dt \right]$

$= \lim_{T\to\infty} \left[v(T)\, e^{-sT} - v(0) + s \int_0^T v(t)\, e^{-st}\, dt \right]$

Assume that $v(T)\, e^{-sT} \to 0$ as $T \to \infty$

$= -v(0) + s \int_0^\infty v(t)\, e^{-st}\, dt = -v(0) + s\,\mathcal{L}(v)$

Hence, $s\,\mathcal{L}(v) - v(0) = -\mathcal{L}(v)$: substitute $v(0) = 2$ and write $V = \mathcal{L}(v)$

$sV(s) - 2 = -V(s)$

$V(s) = \frac{2}{s+1}$

Therefore, $v(t) = \mathcal{L}^{-1}(V) = \mathcal{L}^{-1}\left(\frac{2}{s+1}\right) = 2\,\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$

Lookup table: $\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{at}\Big|_{a=-1} = e^{-t}$

$\Rightarrow \quad v(t) = 2e^{-t}$ is the solution. ✓

Derivatives:

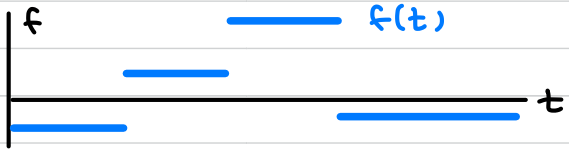
$$\mathcal{L}\left(\frac{dv}{dt}\right) = s\,\mathcal{L}(v) - v(0)$$

$$\mathcal{L}\left(\frac{d^2v}{dt^2}\right) = s^2\,\mathcal{L}(v) - sv(0) - \frac{dv}{dt}(0)$$

→ Laplace transform replaces differentiation with multiplication by s (plus initial conditions)!

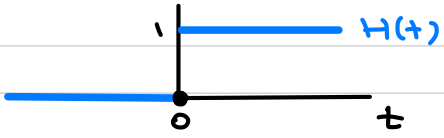
• Step functions

Imagine our forcing function is given by a step function: we switch our forcing on and off



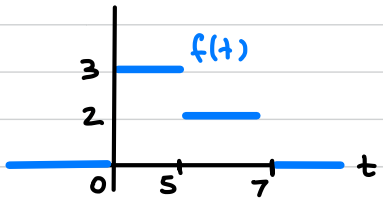
Heaviside function:

$$H(t) := \begin{cases} 0 & t<0 \\ 1 & t\geq 0 \end{cases}$$



Example:

$$f(t) = \begin{cases} 0 & t<0 \\ 3 & 0\leq t<5 \\ 2 & 5\leq t<7 \\ 0 & 7\leq t \end{cases}$$



switch „3" off for $t>5$ — switch „2" on

$$\Rightarrow \quad f(t) = 3H(t) + (-3+2)H(t-5) + (-2+0)H(t-7)$$
$$= 3H(t) - H(t-5) - 2H(t-7)$$

Laplace transform: For $a\geq 0$, we have

$$\mathcal{L}(H(t-a)) = \int_0^\infty H(t-a)\,e^{-st}\,dt = \int_a^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_{t=a}^{t=\infty}$$

need $s>0$ at $t=\infty$

$$= \frac{1}{s}e^{-as} \quad \text{for } s>0$$

• Switching property

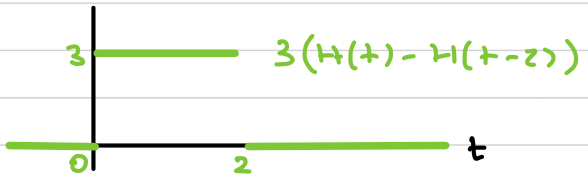
For $a\geq 0$, we have

$$\begin{cases} \mathcal{L}\left(H(t-a)\,f(t-a)\right) = e^{-as}F(s) & \text{where } F(s) = \mathcal{L}(f) \\ \mathcal{L}^{-1}\left(e^{-as}F(s)\right) = H(t-a)\,f(t-a) \end{cases}$$

which can be shown by evaluating the integral that defines the Laplace transform

Example: Solve $\frac{dv}{dt} = -v + 3H(t) - 3H(t-2)$ with $v(0) = 0$

Forcing:



$\rightarrow \quad \mathcal{L}\left(\frac{dv}{dt}\right) = \mathcal{L}(-v + 3H(t) - 3H(t-2))$ with $V = \mathcal{L}(v)$

$s\mathcal{L}(v) - v(0) = -\mathcal{L}(v) + 3\mathcal{L}(H(t)) - 3\mathcal{L}(H(t-2))$ } linearity & derivatives

$sV = -V + \frac{3}{s} - \frac{3}{s}e^{-2s}$ } previous example & initial conditions

Hence, $V(s) = \frac{3}{s(s+1)}(1 - e^{-2s})$

Thus, $v(t) = \mathcal{L}^{-1}\left(\frac{3}{s(s+1)}(1 - e^{-2s})\right)$

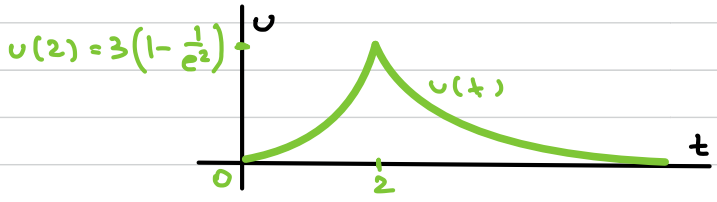
partial fractions: $\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$

$\rightarrow \quad v(t) = 3\mathcal{L}^{-1}\left[\left(\frac{1}{s} - \frac{1}{s+1}\right)(1 - e^{-2s})\right]$

$= 3\left[\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s}e^{-2s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+1}e^{-2s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)\right]$

$= 3\left[1 - H(t-2) + e^{-(t-2)}H(t-2) - e^{-t}\right]$

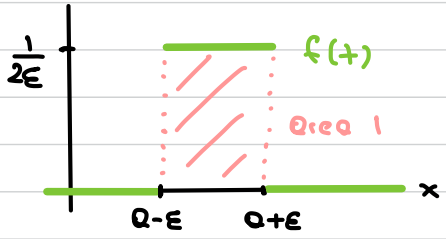
So that $v(t) = \begin{cases} 3(1 - e^{-t}) & 0 \le t < 2 \\ 3(e^{-(t-2)} - e^{-t}) & 2 \le t \end{cases}$



- Impulse forcing

Newton's law describes the motion of a point mass m with velocity $v(t)$ subjected to force $f(t)$: $m\frac{dv}{dt} = f(t)$

Impulse forcing:



$$m\frac{dv}{dt} = f(t) \quad (= 1)$$

$$mv(a+\varepsilon) - mv(a-\varepsilon) = \int_{a-\varepsilon}^{a+\varepsilon} f(t)\,dt$$

$m\,\Delta v = 1$ for all $\varepsilon > 0$

↑ velocity difference

$\Rightarrow$ let $\varepsilon \to 0$ to model a single impulse exactly at $t = a$ that changes mv by one unit

The resulting „generalized function" is denoted by $\delta_a(t)$ (the „δ-function" or „delta-function"). It is characterized by the property that for $a \geq 0$ and any continuous function $g(t)$

$$(1) \quad \int_0^\infty \delta_a(t)\, g(t)\, dt = g(a) \qquad (a \geq 0)$$

Laplace transform of $\delta_a(t)$ for $a \geq 0$:

$$\mathcal{L}(\delta_a) = \int_0^\infty \delta_a(t)\, e^{-st}\, dt \overset{(1)}{=} e^{-as}$$

- Lookup table

$v = \mathcal{L}^{-1}(V)$	$V = \mathcal{L}(v)$
e^{at}	$\frac{1}{s-a}$
$H(t-a)$	$\frac{1}{s}e^{-as}$
$\delta_a(t)$	e^{-as}

Example: $m\frac{dv}{dt} = -v + \delta_a(t)$ with $v(0) = 0$ where $a > 0$

→ apply Laplace transform:

$$\mathcal{L}\left(m\frac{dv}{dt}\right) = \mathcal{L}(-v + \delta_a(t))$$

linearity

$$m\,\mathcal{L}\left(\frac{dv}{dt}\right) = -\mathcal{L}(v) + \mathcal{L}(\delta_a(t))$$

derivatives & lookup table

$$m(sV(s) - v(0)) = -V(s) + e^{-as}$$

$v(0) = 0$

$$msV(s) = -V(s) + e^{-as}$$

$$V(s) = \frac{1}{ms+1}e^{-as} = \frac{1}{m}\frac{1}{s+1/m}e^{-as}$$

$$v(t) = \mathcal{L}^{-1}(V(s)) = \mathcal{L}^{-1}\left(\frac{1}{m}\frac{1}{s+1/m}e^{-as}\right)$$

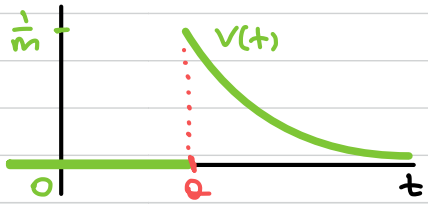
$$= \frac{1}{m}\mathcal{L}^{-1}\left(\frac{1}{s+1/m}e^{-as}\right) = \frac{1}{m}H(t-a)\,\mathcal{L}^{-1}\left(\frac{1}{s+1/m}\right)(t-a)$$

switching property

$$= \frac{1}{m}H(t-a)\,e^{-\frac{1}{m}(t-a)}$$

lookup table

→ $v(t) = \frac{1}{m}H(t-a)\,e^{-\frac{1}{m}(t-a)}$ is the solution to our problem



Additional Material

- Matrix exponentials: $e^A e^B = e^{A+B} \not\Rightarrow AB = BA$
- Second-order equations: repeated roots

Matrix exponentials

Note: $e^A e^B = e^{A+B} \not\Rightarrow AB = BA$

Example:

- $A = \pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \pi \begin{pmatrix} 0 & 10+4\sqrt{6} \\ -10+4\sqrt{6} & 0 \end{pmatrix}$

  $A+B = \pi \begin{pmatrix} 0 & 11+4\sqrt{6} \\ -11+4\sqrt{6} & 0 \end{pmatrix}$

- Let $C = \begin{pmatrix} 0 & a+b \\ -a+b & 0 \end{pmatrix}$ with $a > b \geq 0$, then

  C has eigenvalues $\lambda_{1,2} = \pm i\sqrt{a^2-b^2}$

- Eigenvalues of A, B, $A+B$:

  A: $\lambda_{1,2} = \pm \pi i$
  B: $\lambda_{1,2} = \pm 2\pi i$
  $A+B$: $\lambda_{1,2} = \pm 5\pi i$

- We write each matrix as VDV^{-1} so that its matrix exponential is Ve^DV^{-1}. We have

  A: $e^{D_1} = -1$
  B: $e^{D_2} = 1$
  $A+B$: $e^{D_3} = -1$

  using the expressions for their eigenvalues

  Hence,

  $$e^A = V_1 e^{D_1} V_1^{-1} = -V_1 V_1^{-1} = -1$$
  $$e^B = V_2 e^{D_2} V_2^{-1} = V_2 V_2^{-1} = 1$$
  $$e^{A+B} = V_3 e^{D_3} V_3^{-1} = -V_3 V_3^{-1} = -1$$

  and therefore $e^A e^B = e^{A+B}$ ✓

- $$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & a+b \\ -a+b & 0 \end{pmatrix} = \begin{pmatrix} -a+b & 0 \\ 0 & -a-b \end{pmatrix}$$
  $$\begin{pmatrix} 0 & a+b \\ -a+b & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -a-b & 0 \\ 0 & -a+b \end{pmatrix}$$

  which are <u>not</u> equal unless $b=0$ ✓

Second-order equations: repeated roots

Consider $\frac{d^2u}{dt^2} + p\frac{du}{dt} + qu = 0$. The function $u(t) = e^{\lambda t}$ is a solution if and only if $\lambda^2 + p\lambda + q = 0$, which we can solve to get $\lambda_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$. The two roots will coincide (so that $\lambda_1 = \lambda_2$) if and only if the term under the square root vanishes so that $\frac{p^2}{4} - q = 0$ or $q = \frac{p^2}{4}$. We will now derive the general solution in this case.

Hence, assume $q = \frac{p^2}{4}$ and consider (*) $\frac{d^2u}{dt^2} + p\frac{du}{dt} + \frac{p^2}{4}u = 0$

We cast (*) as the first-order system

(**) $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -p^2/4 & -p \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ with eigenvalues $\lambda = \lambda_1 = \lambda_2 = -\frac{p}{2}$.

Next, we write

$$A = \begin{pmatrix} 0 & 1 \\ -p^2/4 & -p \end{pmatrix} = -\frac{p}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \underbrace{\begin{pmatrix} p/2 & 1 \\ -p^2/4 & -p/2 \end{pmatrix}}_{=:B} = -\frac{p}{2}1 + B$$

Note that $B^2 = \begin{pmatrix} p/2 & 1 \\ -p^2/4 & -p/2 \end{pmatrix}\begin{pmatrix} p/2 & 1 \\ -p^2/4 & -p/2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

and therefore $e^{Bt} = 1 + tB = \begin{pmatrix} 1 + \frac{p}{2}t & t \\ -\frac{p^2}{4}t & 1 - \frac{p}{2}t \end{pmatrix}$

Since $A = -\frac{p}{2}1 + B$ and $1B = B1$, we have

$$e^{At} = e^{-\frac{p}{2}t1}e^{Bt} = e^{-\frac{p}{2}t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+p/2\,t & t \\ -\frac{p^2}{4}t & 1-\frac{p}{2}t \end{pmatrix}$$

$$= e^{-\frac{p}{2}t}\begin{pmatrix} 1+p/2\,t & t \\ -\frac{p^2}{4}t & 1-\frac{p}{2}t \end{pmatrix} = e^{\lambda t}\begin{pmatrix} 1-\lambda t & t \\ -\lambda^2 t & 1+\lambda t \end{pmatrix}$$

Since $\begin{pmatrix} u \\ v \end{pmatrix}(t) = e^{At}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ is the general solution of (**), we extract the first component to get

$$u(t) = e^{\lambda t}\left[(1-\lambda t)u_0 + t v_0\right] = \underbrace{e^{\lambda t}c_1 + t e^{\lambda t}c_2}_{\text{general solution of (*)}}$$

with $c_1 = u_0$ and $c_2 = v_0 - \lambda u_0$.